Teaching the representations of concepts in Calculus: The case of the Intermediate Value Theorem

Article *in* PRIMUS: problems, resources, and issues in mathematics undergraduate studies · October 2018

DOI: 10.1080/10511970.2018.1540023

3 authors, including:

Erika J. David

Arizona State University

Morgan Sellers

Arizona State University

PREPRINT VERSION
In press in *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*

Teaching the Representations of Concepts in Calculus: The Case of the Intermediate Value Theorem

Erika J. David **Kyeong Hah Roh** **Morgan E. Sellers**

Arizona State University

Abstract: This paper offers instructional interventions designed to support undergraduate math students' understanding of two forms of representations of Calculus concepts, mathematical language and graphs. We first discuss issues in students' understanding of mathematical language and graphs related to Calculus concepts. Then, we describe tasks, which are situated in the context of the Intermediate Value Theorem, that are designed to promote students' understanding of multiple quantifiers in mathematical statements, as well as outputs and points on graphs of functions in the Cartesian plane. We offer suggestions for instruction and include sample dialogue to illustrate how these tasks may be used in the classroom.

Keywords: Teaching and Learning Calculus, Representations, Quantifiers, Graphs of Functions, Intermediate Value Theorem

1. INTRODUCTION

Many undergraduate students enroll in a Calculus course at some point in their program to fulfill requirements for degrees not only in mathematics, but also in business, life sciences, engineering, and other mathematically-situated fields [7]. Mathematics education researchers have reported student difficulty with concepts at the heart of Calculus, including limits, continuity, derivatives, and integrals [13, 18, 19, 21, 22]. To better support students in understanding these ideas, researchers and mathematicians have developed instructional interventions for classroom teaching of these concepts [9, 11, 20]. In our experience, students often face difficulties not only with understanding and applying the concepts in Calculus, but also with understanding the ways in which these concepts are presented in textbooks. For instance, textbooks often provide mathematical statements and graphical illustrations when presenting the definitions and theorems central to Calculus. The intent of both mathematical statements and accompanying graphical illustrations is to support students' understanding of the relevant concept. However, research suggests students may not interpret mathematical language or the graphical representations found throughout Calculus in the ways intended [3, 4, 5, 8, 12, 15, 16]. Although students may understand aspects of a mathematical concept, some of them may have difficulty in expressing the concept with mathematical representations [10, 14]. From our perspective, the understanding of the representations of concepts in Calculus and the concepts themselves are inherently intertwined. Furthermore, students' understanding of the representations of concepts in Calculus may promote their understanding of the mathematical concepts and vice versa.

In this paper, we propose instructional interventions aimed at supporting students in understanding two modes of representations of Calculus concepts, *mathematical language* and *graphical representations*. We use the Intermediate Value Theorem (IVT) as the context for the tasks in our interventions, as the IVT is a representative example of a Calculus concept that is presented through both language and graphs. The mathematical language of the IVT, which includes 'if-then' structure and multiple quantifiers, is commonly found in other statements from Calculus. Also, like other statements from Calculus, it is often accompanied by a graphical representation. Although we view our interventions as relevant to Calculus courses, we anticipate that instructors of more advanced mathematics courses may find them useful as well.

While we situate our discussion in the context of the IVT, the goal of this paper is not to illustrate how to teach students to use or prove the IVT. Rather, we use the IVT in this paper as an example of how to teach the mathematical language and graphical representations used to present the concepts of Calculus. First, we share samples from our observations of issues in students' understanding of these two representations of the IVT, not simply to point out student errors, but in order to highlight possible student issues for instructors to be aware of. We then present tasks for teaching the mathematical language and graphical representations found in undergraduate Calculus courses using the IVT, specifically designed to address the issues we observed. We include suggestions for instructors to employ, modify, or extend these tasks in the classroom. Finally, we conclude with a discussion of the implications of teaching the representations of Calculus concepts.

2. ISSUES IN STUDENTS' UNDERSTANDING OF REPRESENTATIONS OF THE INTERMEDIATE VALUE THEOREM

The Intermediate Value Theorem (IVT) is one of the central mathematical statements introduced to students in Calculus textbooks [e.g., 6, 17]. One such version of the statement is as follows:

> **The Intermediate Value Theorem.** Suppose that f is a continuous function on $[a, b]$ such that $f(a)\neq f(b)$. Then for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$ (adapted from [17]).

Some textbooks capture this idea succinctly as follows: A continuous function f on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$ (e.g., adapted from [6]). Often, the IVT statements are also accompanied by graphical representations like the ones in Figure 1 [e. g., 17].

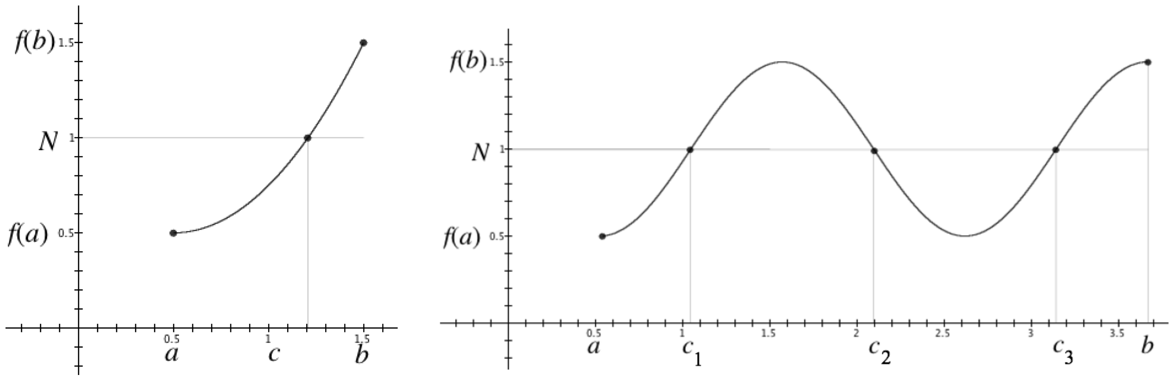


Figure 1. Possible graphs illustrating the IVT [3, p. 93]

The graph in Figure 1 (left) illustrates that for an output value, N, between $f(a)$ and $f(b)$, there is a corresponding value of the input, c, between a and b, such that $f(c) = N$. In the graph in Figure 1 (right), a value for N is shown with multiple corresponding input values for c, c_1, c_2, c_3, such that $f(c)=N$. Together with the theorem statement, these graphs can be used to help illustrate that for continuous functions, for each output value in a certain range, the IVT guarantees the existence of at least one corresponding input value, resulting in the output value.

In recent research that we have conducted, we found that the representations of the IVT played a crucial role in students' understanding of the theorem. Specifically, students struggled with the mathematical language of the IVT statement, including the multiple quantifiers it contains [16]. When explaining the meaning of the IVT, some students used graphical representations in unconventional ways, which often led them to incorrectly interpret the statement [3].

In the following sections, we discuss in more detail the issues we observed in students' interpretation of mathematical language and graphical representations of the IVT. We note that in our descriptions below, our primary purpose is not to evaluate student thinking, but to try to understand where and how student thinking may differ from mathematical convention. In this regard, we use normative understandings as a reference point to understand how students make sense of mathematical language and graphical representations.

2.1 Issues in Students' Understanding of Mathematical Language

In our research [16], we conducted clinical interviews [2] in which we asked nine undergraduate students who had completed at least Calculus I to evaluate and explain what the IVT statement means in their own words, as stated above. We did not tell the students that this statement was the IVT. While most students correctly evaluated the statement as true, we found that many of these students' explanations of the meaning of the statement were not compatible with our understanding of the theorem as previously described. For example, one student, Mike, who had completed not only Calculus I, but also a Transition-to-Proof course, re-ordered the quantifiers in his explanation of the IVT. He claimed: "it is stating that there exists a real number c between the interval (a, b) such that a function at that number is all the real numbers." Mike interpreted the IVT statement as if it claimed the existence of a value c for all outputs $[N]$. Because Mike reversed the order of the quantifiers (there exists, for all) from the given IVT statement, he concluded that the statement was false. While Mike reversed the order of the quantifiers but maintained the attached variables (c, N) to each quantifier, we observed other students who reversed both the order of the quantifiers and variables in the IVT. Our findings align with previous research indicating that students may switch the order of the variables [12], as well as switch the order of quantifiers in mathematical statements [5].

Given the complexity of the language in statements like the IVT, it is understandable why students may confuse the order of the quantifiers or variables, especially since most Calculus students receive no formal instruction on the meaning of quantifiers explicitly when the IVT is introduced in Calculus. Additionally, the order of the quantifiers in a statement communicates important relationships between the variables involved in the statement. For instance, in the IVT statement, the mathematical language 'for all N, there exists c' suggests that the value(s) of c depends on the selection of N, although such a dependence of c on N is not explicitly stated in the wording of the statement. The effects of how students understand the mathematical language used to describe concepts like the IVT extend beyond Calculus. The way in which students interpret the order of quantifiers or variables in mathematical statements may impact how they apply statements like the IVT in context or write proofs of these statements. Thus, from our perspective, students' understanding of mathematical language is vital to their understanding of the concepts of Calculus, which provide a foundation for advanced mathematics.

2.2 Issues with Students' Understanding of Graphical Representations

In our research [3], we also observed students who interpreted aspects of graphical representations in unconventional ways, which in turn led to unconventional interpretations of the IVT. Nate, who had completed Calculus, and recently completed an Introductory Analysis course, was one such student. Nate correctly interpreted the order of the quantifiers and variables in the IVT. However, when he explained the IVT with a graph, we found that he interpreted the meaning of the statement unconventionally due to his meaning for points on the graph. Specifically, we discovered that Nate interpreted outputs of the function as points on the graph, rather than as output values on the y-axis. He described N's between $f(a)$ and $f(b)$ and labeled a point as 'N' whose outputs values were outside the range of outputs between $f(a)$ and $f(b)$ (see Figure 2.). Due to his understanding of N as a point on the graph, Nate explained that the phrase 'N between $f(a)$ and $f(b)$' in the IVT statement referred to all points along the graph, rather than a limited range of output values on the y-axis. Nate's unconventional interpretation of points on the graph of a function affected his understanding of the IVT statement. In fact, Nate's interpretation of the IVT as guaranteeing the existence of a value for c for each point on the graph between the endpoints does not depend on the continuity of the function and is trivially true.

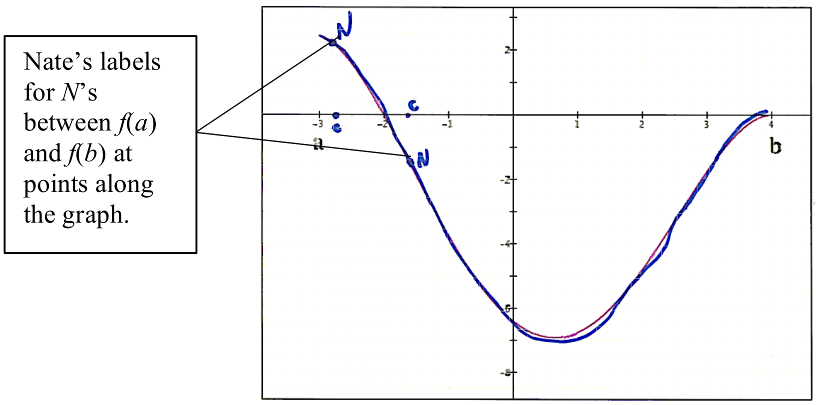


Figure 2. Nate's labels of possible N's between $f(a)$ and $f(b)$

Nate was not the only student who interpreted N's between $f(a)$ and $f(b)$ not as values, but as points on the graph. In fact, almost half of the students whom we interviewed, including Nate, indicated during some portion of the interview that they thought about points on graphs as outputs of functions, rather than as ordered pairs of input and output values of functions [3]. Interpreting points on graphical representations in this way led students to misunderstand the statements we presented and, consequently, to evaluate some of the statements incorrectly. In addition to influencing students' understanding of concepts like the IVT, we anticipate that their understanding of graphs would impact their comprehension of mathematical statements, their abilities to prove theorems, and other activities in advanced mathematics courses that rely heavily on graphical representations of functions. Graphs have been shown to be a powerful support for students [1], and may assist them in better understanding the distinctions in mathematical language. However, in order for graphs to be a powerful support, students need to interpret them in line with mathematical convention.

3. TASKS & SUGGESTIONS FOR INSTRUCTION

The tasks we present in this paper are designed to actively engage students in experiencing issues in their understanding of mathematical language and graphical representations firsthand. As such, we do not expect that all students will work through these tasks with ease and accuracy. Instead, we intend for these tasks to give students the opportunity to experience cognitive conflict [18]. In our view, students may experience cognitive conflict regarding a concept when they encounter a situation which, from their perspective, challenges their current understanding of that concept. In order to resolve this conflict between their own understanding and the situation at hand, the student may adjust their understanding to account for the new situation, by reflecting on their own thinking. Thus, experiencing and resolving cognitive conflict could promote student learning.

In this section, we present Tasks 0-3 designed to support classroom discourse around the mathematical language and graphical representations of concepts in Calculus. Along with each task, we include suggestions for how an instructor may use them in the classroom. Table 1, below, offers an overview of each task, including a description of the purpose of each task and materials used with the task.

Table 1. Outline of Tasks

Task 0: Presentation and Evaluation of Statements Purpose: Familiarize students with the four statements used throughout the intervention Materials: Statements (Table 2)
Task 1: Discussion of Mathematical Language

Task 1.1:	Students Compare Meaning of the Statements
	Purpose: Orient students to the meaning of quantifier, variable order in the four statements
	Materials: Statements (Table 2), Polling Method
Task 1.2:	Students Discuss Meaning of Order of Quantifiers and Variables
	Purpose: Direct students' attention to the role of quantifier, variable order in the statements
	Materials: Statements (Table 2), Sample Student Responses (Table 3)
Task 2: Discussion of Graphical Representations	
Task 2.0:	Students Create Graphs to Explain Statement 1
	Purpose: Uncover students' current meanings for outputs, points on graphs of various functions
	Materials: Statement 1 (Table 2)
Task 2.1:	Students Discuss Meaning of Points on Cartesian Graph
	Purpose: Orient students to the meaning for outputs, points on the graphs of functions, expose students to non-monotone graph
	Materials: Sample Student Responses & Labeled Graphs (Table 4), Graph (Figure 3)
Task 2.2:	Instruction on Cartesian Plane Graphs
	Purpose: Provide instruction on conventions of graphing in Cartesian coordinates
	Materials: Labeled Graph (Figure 4)
Task 3: Re-evaluation of Statements with Graphs	
	Purpose: Provide opportunity for students to apply their (potentially) refined meanings for mathematical language and graphical representations of functions
	Materials: Statements (Table 2) & Unlabeled Graphs (Figure 1 & 4)

For our purposes, these tasks are designed to create situations in which cognitive conflict may arise for students relative to their understanding of mathematical language and graphs. In addition, these tasks may be used by an instructor in class to reveal students' issues with these representations, similar to the issues we observed as described in Section 2. Although these tasks alone may not provide a resolution for these issues, they might bring them to light for discussion among students and the instructor, which may facilitate their resolution. While we provide these tasks in a given order, we recognize that instructors may wish to use the tasks flexibly to meet the needs of their students.

3.0 Task 0: Presentation and Evaluation of Statements

In Task 0, students read and evaluate the four statements in Table 2.

Table 2. IVT and Similar Statements

Statement 1	Suppose f is a continuous function on $[a, b]$ and that $f(a)\neq f(b)$. Then for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c)=N$.

Statement 2 (IVT)	Suppose f is a continuous function on $[a, b]$ and that $f(a)\neq f(b)$. Then for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c)=N$.
Statement 3	Suppose f is a continuous function on $[a, b]$ and that $f(a)\neq f(b)$. Then there exists a real number N between $f(a)$ and $f(b)$, such that for all real numbers c in (a, b), $f(c)=N$.
Statement 4	Suppose f is a continuous function on $[a, b]$ and that $f(a)\neq f(b)$. Then there exists a real number c in (a, b), such that for all real numbers N between $f(a)$ and $f(b)$, $f(c)=N$.

Typically, Statement 2 (the IVT) alone is introduced to students in Calculus courses. However, students may not recognize the necessity of the particular ordering of the quantifiers and variables describing the IVT (see Mike's case in the section 2.1). Therefore, we suggest presenting the IVT along with other statements that are similar to the IVT, but contain a different order of quantifiers or variables, as shown in Table 2. In Table 2, Statement 2 is the IVT, Statement 4 contains a reordering of the quantifiers, Statement 1 contains a reordering of the variables, and Statement 3 has both. Each of these statements has a distinct meaning, given the particular arrangement of the quantifiers and variables. Providing each of these combinations of quantifiers and variables may give students the opportunity to reflect on the differences in semantic meaning in the statements, which they may not have had previously.

While using this task, we recommend that an instructor not tell students that the IVT is among the statements, so students rely on their own thinking rather than external authority. Asking the students to evaluate the statements invites students to carefully consider the meaning of each of the statements and may promote a comparison of the statements. Depending on students' prior experience to reading mathematical language, some students may have more difficulty accessing the differences among the statements without additional resources. In this case, we suggest asking students to graph functions or to provide graphs of functions to help them think about the statements, which we include in Tasks 2 and 3. Once students have had sufficient time to familiarize themselves with the statements and evaluate each one, the instructor may proceed to Task 1.

3.1 Task 1: Discussion of Mathematical Language

Task 1 is designed to draw students' attention to the order of the quantifiers and the variables in the statements in Table 2. Task 1 is divided into two parts. In Task 1.1, students compare the meaning of the statements and group statements according to which they believe have the same meaning. In Task 1.2, students discuss the meaning of the quantifier and variable order by comparing and evaluating sample student responses regarding the meaning of the statements.

3.1.1 Task 1.1: Students Compare Meaning of the Statements

In Task 1.1, students compare the meanings of each of the statements in Table 2 to decide whether they think any of the statements are the same in meaning. We suggest that an instructor poll students' responses in the class, using clickers or a show-of-hands, from the following choices:

A) All four statements are the same in meaning.
B) Statements 1 and 2 are the same and Statements 3 and 4 are the same.
C) Statements 1 and 3 are the same and Statements 2 and 4 are the same.
D) Statements 1 and 4 are the same and Statements 2 and 3 are the same.
E) All four statements are different in meaning.

These five choices capture many of the ways students may compare the meaning of the four statements, as we observed in our study [16]. Once students have had time to consider and compare the meanings of the four statements and respond to the question above, the instructor may proceed to Task 1.2.

3.1.2 Task 1.2: Students Discuss Meaning of Order of Quantifiers and Variables

The results of the poll in Task 1.1 may indicate possible issues students may have with understanding why the order of the quantifiers or variables matters. Table 3 includes some sample student responses for choices A-E from the poll regarding the meaning of the four statements, which we created based on real student responses from our previous research [16].

Table 3. Sample Student Reasoning for Selecting A–E in Task 1

Sample Student Responses	Characteristics of Student Responses	
Anees: *To me, all four statements mean the same thing. They are all saying for continuous functions, the input is c and the output is N in the equation 'f(c)=N.'*	**No attention to quantifiers.** Anees' response does not contain quantifier words such as "for all" and "there exists." Rather than attending to quantifiers, Anees' response indicates an attention only to the input-output relationship of c and N in the equation '$f(c)=N$.' Anees' response may be typical for students with limited exposure to quantified statements.	S1: $\forall c \exists N$ S2: $\forall N \exists c$ S3: $\exists N \forall c$ S4: $\exists c \forall N$
Basil: *Statements 1 and 2 both mean the same thing and statements 3 and 4 mean the same thing. Statements 1 and 2 both have "for all" first and statements 3 and 4 both have "there exists" first.*	**Distinction in quantifier order but not variable order.** Unlike Anees, Basil recognizes that "for all" appearing first in the statement has a distinct meaning from "there exists" appearing first. However, his response does not contain a distinction in the order of the variables, N and c.	S1: $\forall c \exists N$ S2: $\forall N \exists c$ S3: $\exists N \forall c$ S4: $\exists c \forall N$
Carlos: *Statements 1 and 3 are the same and statements 2 and 4 are the same. Statement 1 says for all c, there exists an N and Statement 3 says there exists an N for all c. Saying "for all inputs, c, you can find an output, N" is the same as saying "there is an output N for all inputs c."*	**No distinction in quantifier order but keeps attached variable.** Carlos thinks reversing the order of the quantifiers has no effect on the meaning of the statements. Carlos does maintain the attachment of the variables to the quantifiers.	S1: $\forall c \exists N$ S2: $\forall N \exists c$ S3: $\exists N \forall c$ S4: $\exists c \forall N$
Dalia: *Statements 1 and 4 are the same and statements 2 and 3 are the same. I see that c comes before N in statements 1 and 4 and that N comes before c in Statements 2 and 3.*	**Distinction in variable order but not quantifier order.** While Dalia recognizes that c appearing before N means something different than N appearing before c, she does not attend to the order of the quantifiers.	S1: $\forall c \exists N$ S2: $\forall N \exists c$ S3: $\exists N \forall c$ S4: $\exists c \forall N$

Edith: *I think all four statements are saying different things. The order of the words "for all" and "there exists" and the order of the variables c and N are different which makes the statements have different meanings.*	**Distinction in quantifier and variable order.** Edith claims that both the order of "for all" and "there exists" as well as the order of *c* and *N* make all four statements different in meaning.	S1: $\forall c \exists N$ S2: $\forall N \exists c$ S3: $\exists N \forall c$ S4: $\exists c \forall N$

These sample responses in Table 3 point out different aspects of the meaning of the order of quantifiers or variables, and thus may be used to facilitate a discussion of the meaning of the mathematical language of the statements. For instance, based on the results of the poll, an instructor may choose to discuss the sample student responses corresponding to highly-selected choices. Presenting students with a response related to a commonly chosen response may help students to articulate the reasons they chose the answers they did and be a means to discuss their reasoning more directly.

Regardless of the incorrect response or responses chosen for discussion, we would suggest that an instructor always include response E in the discussion. Edith's response may serve two purposes. First, for students who have not considered the order of the quantifiers or variables at all, Edith's response may bring this issue to light for consideration. Second, her response may lead to a discussion that focuses on why the orders of the quantifiers and the variables matter in the statements. While the sample student responses we provide in Table 3 may represent some students' thinking, we recognize that students may choose A–E for many reasons. These responses are not meant to exhaust all possible ways students may reason about the statements.

Through discussing these possible responses about the meaning of these four statements, the ordering of the quantifiers and variables in mathematical statements may be brought to light for students. Once the issues of the meaning of the order of quantifiers and variables have been brought up, an instructor may proceed to Task 2. While students' attention may be drawn to such issues of language after working with Task 1, they may be unsure of how to resolve these issues. Working through Tasks 2 and 3 may provide more opportunity for students to continue to reflect and ultimately resolve conflicts they may be experiencing relative to the mathematical language of the statements.

3.2 Task 2: Discussion of Graphical Representations

When examining the four statements in Table 2, some students might interpret graphical representations differently than mathematical convention. Students' unconventional ways of understanding graphs influences their understanding of the IVT and similar statements, as we described in Section 2.2. The goal of Task 2 is to orient students to using various graphs, as well as to discuss how to properly interpret outputs and points on graphs of continuous functions alongside the four statements in Table 2. In our study, the way in which students interpreted the mathematical language influenced how they understood and used graphs. For this reason, we suggest using this task following Task 1, so the issue of mathematical language has been raised for students already before Task 2.

We split Task 2 into 3 parts: in Task 2.0, students create their own graphs, in Task 2.1, students consider alternate meanings for outputs and points in graphical representations with sample student responses, and in Task 2.2, students receive instruction on the mathematical meaning of points in the Cartesian plane.

3.2.0 Task 2.0: Students Create Graphs to Explain Statement 1

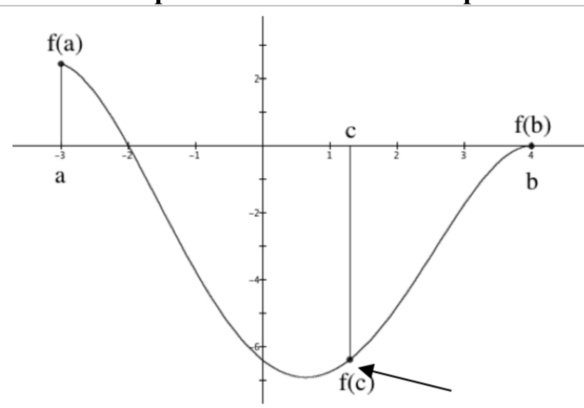
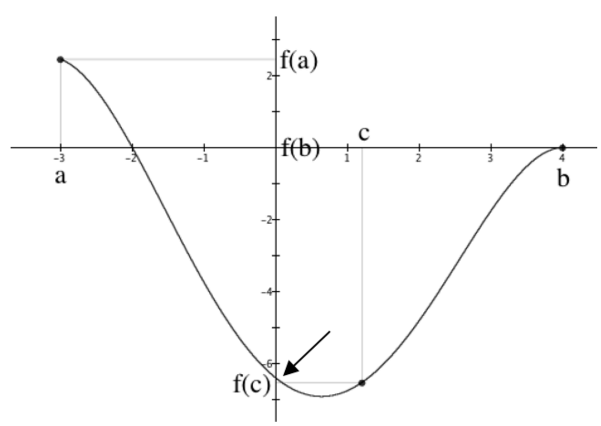
In the first phase of Task 2, students create graphs of various functions to describe their understanding of Statement 1. We suggest only focusing on Statement 1 for this portion of the task, as the way in which students interpret points on a graph likely impacts their evaluation of this statement, as we observed with Nate (Section 2.2). Furthermore, unlike Statements 2, 3, and 4, the conclusion of Statement 1 holds true for some continuous function, while for others the conclusion of Statement 1 is false. For example, the conclusion of Statement 1 is true for an increasing or decreasing function (Figure 1 left), or a periodic function in which the range of outputs does not exceed the bounds at the endpoints (Figure 1 right). However, the conclusion of Statement 1 is false for a function in which the range exceeds the values between $f(a)$ and $f(b)$ (see graph in Table 4). Thus, while Statement 1 is false, students may work with multiple graphs and arrive at different conclusions regarding the evaluation of Statement 1.

An instructor may encourage students to label the relevant values and points on their graphs, such as a, b, c, $f(a)$, $f(b)$, $f(c)$, and N. If needed, an instructor might remind students of the hypothesis of Statement 1, namely that their graphs should be of continuous functions with $f(a)$ not equal to $f(b)$. Once students have drawn a graph of their own, students may take turns explaining their evaluation of Statement 1 with the graph that they drew. The instructor may make note of the label placement on students' graphs, as this may indicate how students are interpreting points and outputs on their graphs. For instance, if a student places output labels at points along the graph rather than the y-axis, this may indicate that the student is thinking of points as solely outputs, like Nate did in Section 2.2. After students have created and shared their graphs, the instructor may present several graphs to the class for more opportunities for students to consider various graphs (e.g., graphs in Figure 1 and graph in Table 4). Once students have considered several types of graphs with Statement 1, the instructor may proceed to Task 2.1.

3.2.1 Task 2.1: Students Discuss Meaning of Points on Cartesian Graph

Some students may consider points as a pair of input and output values while others may consider points as only output values, as described in Section 2.2. Thus, the goal of Task 2.1 is for students to discuss which meaning for outputs of functions and points on the graphs of functions is relevant to the four statements in Table 2. In Task 2.1, students are presented with the two sample student responses we provide in Table 4. The two student responses contrast the two interpretations of outputs and points on a graph discussed in Section 2.2. Laurel's response in Table 4 indicates that she considers outputs to be locations of points on the graph and points to be outputs. In contrast, Victor's response in Table 4 indicates that he interprets outputs to be values on the *y*-axis and points as ordered pairs of input and output values.

Table 4. Sample Student Responses and Accompanying Graphs

Sample Student Responses	Sample Student Labeled Graphs
Laurel: *Here's f(a) and here's f(b), at these points on the graph. If I pick f(c), which is a point on the graph, it will be between f(a) and f(b).*	
Victor: *Here's f(a) and here's f(b) on the y-axis. If I pick f(c), which is an output here on the y-axis, it will not be between f(a) and f(b).*	

As our research indicates (see Section 2.2), some students may align themselves with Laurel's response, and others may agree with Victor's response in their interpretation of outputs and points on graphs of functions. Laurel places output labels at points on the graph, and she explains that $f(c)$ is between the points labeled $f(a)$ and $f(b)$. She interprets outputs of the function as points on the graph, rather than as values located on the y-axis. With reasoning similar to Nate (see Section 2.2), Laurel explains for the c that she selects, the $f(c)$ she labels is between $f(a)$ and $f(b)$ because $f(c)$ is a point on the graph between the endpoints on the graph of the function. For Victor, however, $f(c)$ is not between $f(a)$ and $f(b)$ because it is not in the range of values between the values $f(a)$ and $f(b)$ on the y-axis. Victor labels the graph with the output labels on the output axis, with the value of $f(c)$ falling outside the range of values between the values of $f(a)$ and $f(b)$. Victor explains that he thinks for the c that he selects, the $f(c)$ he labels is not a value between $f(a)$ and $f(b)$.

An instructor may facilitate discussion through questions contrasting Laurel's response and Victor's response allowing students to discuss the differences in each of these graph labels and how each student is considering points on the graph. One possible discussion is as follows:

Instructor:	What is the difference in where the two students placed their label for $f(c)$?
Student 1:	Laurel put $f(c)$ next to the point she picked.
Student 2:	Yeah, and Victor put $f(c)$ on the y-axis in line with that point.
Instructor:	Yes, notice that Laurel placed the label $f(c)$ at a point on the graph, but Victor placed the label $f(c)$ on the y-axis.
Student 3:	Does it really matter?

Following this discussion regarding how each student is interpreting points on the graph, an instructor may also discuss how each student is thinking about whether $f(c)$ is between $f(a)$ and $f(b)$. An instructor may continue the conversation by presenting the image in Figure 3 and asking students whether the output at each of the indicated points is between $f(a)$ and $f(b)$. For students who consider outputs to be points on the graph, they may respond that the output at point (1) *is* between $f(a)$ and $f(b)$ while the output at point (2) is not. However, students who consider output values at these points to be the y-coordinate may respond in the opposite way. An instructor may ask students to justify their reasoning regarding whether the outputs at these two points are between $f(a)$ and $f(b)$.

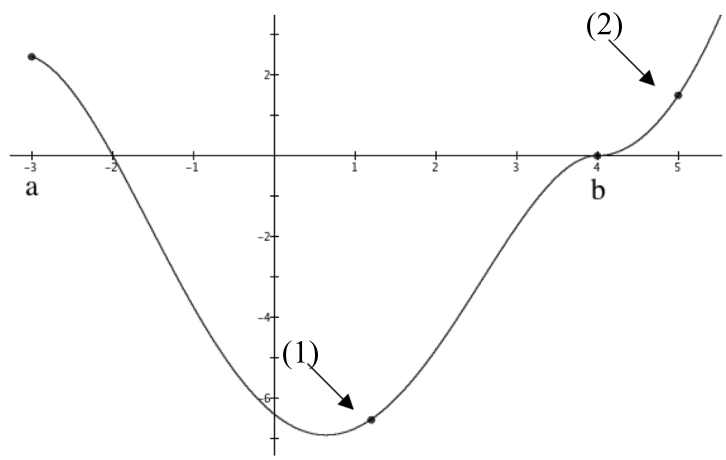


Figure 3. Graph to discuss between $f(a)$ and $f(b)$

Through discussing Laurel's and Victor's meaning for points on a graph and whether the outputs at the points in Figure 3 are between $f(a)$ and $f(b)$, students may reflect on their meanings for outputs and points on a graph. In discussing these two prompts, students who had been thinking about $f(c)$ as a point on the graph, rather than as a value, may begin to consider the alternative that $f(c)$ represents a value represented on the output axis. Once the distinction in these two possible interpretations of points on a graph has been raised, even if it has not been resolved for some students, an instructor may proceed to the final phase of this task.

3.2.2 Task 2.2: Instruction on Cartesian plane graphs

After discussing the two sample student responses in Table 4, some students may have questions about the conventions of graphing in the Cartesian plane. In particular, some students may still fail to distinguish points on the graph of a function and outputs of the function, or think of outputs as *points*, rather than points as representing a pair of input and output *values*. Because graphing points in the Cartesian plane follows a mathematical convention, an instructor may wish to propose the following image with labels for the input value, output value, and point as an ordered pair.

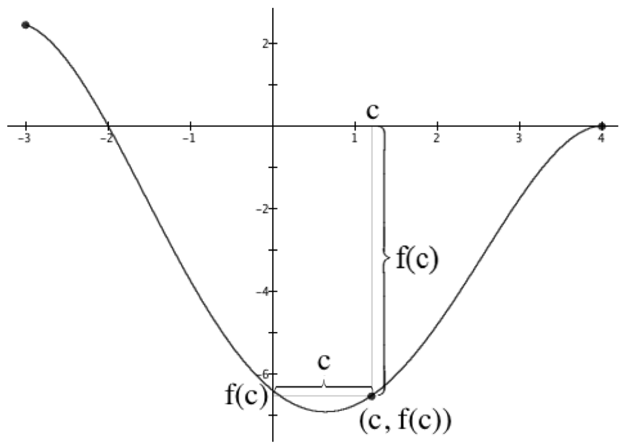


Figure 4. Diagram of point on a graph as ordered pair

The emphasis of this diagram is that a point on the graph of a function represents the coordination of two values simultaneously, rather than just one. The location of a point in the Cartesian plane is found by locating the point that is, in this case, c units to the right of the origin and $f(c)$ units above the origin. Then, $f(c)$ represents an output value along the y-axis, not the location of the point in the Cartesian plane. Thus, the coordinate $(c, f(c))$ represents the location of the point, uniting two values, one of the input and one of the output. After students have seen this diagram, they may revisit the sample responses from Laurel and Victor in 3.2.2 to decide whom they agree with. By the end of this phase, students should view Victor's reasoning as aligned with mathematical convention, whereas Laurel's reasoning is not.

3.3 Task 3: Re-evaluation of Statements with Graphs

Now that students have been exposed to different possible interpretations of mathematical language, and are aware of the conventions of graphical representations of functions, they may benefit from re-evaluating the statements. We suggest presenting the four statements in Table 2 again to students, along with the unlabeled versions of three graphs, the two from Figure 1 and the graph in Figure 4. Together, these three graphs represent a spectrum of possible continuous functions including a monotone increasing function (e.g., Figure 1, left), a function that is not one-to-one (e.g., Figure 1, right), and a function whose output values exceed the range of values between the endpoints' output values (e.g., Figure 4).

An instructor may ask students to re-evaluate each of the four statements in Table 2, using the graphs as examples of continuous functions to consider. Following Tasks 1 and 2, students may have changed the way that they think about the meaning of the order of the quantifiers and variables, or the meaning of points on the graph of a function. Thus, their evaluation of the four statements may change due to their updated meanings for the mathematical language or graphical representations associated with the statements. After students have re-evaluated the statements, instructors may ask students to work in pairs to explain to each other their evaluations of each of the statements using the three graphs. Following the discussions in pairs, an instructor may choose to re-poll students using the four options from Task 1 to check if they think any of the statements are the same in meaning. Through re-visiting the statements and evaluating them using graphs as examples, students may further reflect on their meanings for the mathematical language used in each of the statements, as well as their interpretation of outputs and points on the graphs. The process of explaining their reasoning aloud with sample graphs may encourage students to further consider the role of different types of quantifiers ('for all' and 'there exists'), the order of the quantifiers, and the order of the variables in the statements.

4. DISCUSSION

Undergraduate Calculus curricula typically present students with many definitions and theorems regarding the properties of continuous functions, described using mathematical language and often represented with graphs. In teaching undergraduate Calculus, instructors may seek to provide additional support to their students in understanding the language and graphical representations. Findings from our research have revealed potential issues in students making sense of statements about real valued functions including the IVT [16]. The second issue we observed involved students' unconventional interpretation of points on graphs of real valued functions [3]. The instructional interventions we provide in this paper are designed to combat the issues we observed in ways that would allow students to strengthen their meanings for representations first-hand. Through working with statements with differences in quantifier order and variable order in Table 2, students would have the opportunity to reflect on the impact these differences in mathematical language have on the meaning of a statement. Additionally, working with a pair of contrasting graph labels in Table 4 would provide students with the opportunity to reflect on their own meaning for outputs and points on a graph, and thus how they interpret graphs of functions related to statements from Calculus.

While the tasks presented in this paper are primarily described to be used in order over several class periods, we envision multiple successful ways instructors may choose to implement these tasks in their classes. For instance, an instructor may choose to assign Task 0 to students as homework to prepare for Tasks 1-3 in class. Another instructor may wish to pair Task 0 with Task 2.0 followed by Task 3 during one class period, and then return to Task 1 and the remainder of Task 2 the following class. In either case, students are given the opportunity to encounter differences in mathematical language and graphical interpretations firsthand, which is the overall purpose of these tasks as a whole.

Supporting students in understanding representations of the IVT, both language and graphs, may also support students in understanding other statements from Calculus with similar structure, such as the Mean Value Theorem, Rolle's Theorem, and the Extreme Value Theorem. In this paper, we suggest ways these tasks may be used in Calculus classes to raise and resolve the issues that this paper addressed. We also contend that supporting students in understanding the language and visual representations of mathematical statements throughout Calculus may improve their understanding of the concepts they communicate, as well as strengthen their abilities in proof and argumentation in more advanced courses.

REFERENCES

[1] Arcavi, A. 2003. The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics.* 52(3): 215-241.

[2] Clement, J. 2000. Analysis of clinical interviews: Foundations and model viability. In R. Lesh & A. Kelly (Eds.), *Handbook of research methodologies for science and mathematics education* (pp. 547-589). Hillsdale, New Jersey: Lawrence Erlbaum.

[3] David, E. J., Roh, K. H., and Sellers, M. E. in press. Value-thinking and location-thinking: Two ways students visualize points and think about graphs. *Journal of Mathematical Behavior.*

[4] Dawkins, P., and Roh, K. H. 2016. Promoting metalinguistic and metamathematical reasoning in proof-oriented mathematics courses: a method and a framework. *International Journal of Research on Undergraduate Mathematics Education,* 2:197–222.

[5] Dubinsky, E., and Yiparaki, O. 2000. On student understanding of AE and EA quantification. In Dubinsky, E., A. H. Schoenfeld, and J. Kaput (Eds.), *CMBS issues in mathematics education* (pp. 239-289). Providence, RI: American Mathematical Society.

[6] Finney, R.L., and Thomas, G. B. 1994. *Calculus*. (2nd ed.). Reading, MA: Addison-Wesley Publications.

[7] Kirkman E. 2013. CBMS 2010 survey. Providence, RI: American Mathematical Society.

[8] Moore, K. C., and Thompson, P. W. 2015. Shape thinking and students' graphing activity. In Fukawa-Connelly, T., N. E. Infante, K. Keene and M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). Pittsburgh, PA: RUME.

[9] Oehrtman, M. 2008. Layers of abstraction: Theory and design for the instruction of limit concepts. In Carlson, M. and C. Rasmussen (Eds.), *Making the Connection: Research and Practice in Undergraduate Mathematics*, *MAA Notes* Volume. (Vol. 73, pp. 65-80).

[10] Oehrtman, M., Swinyard, C., and Martin, J. 2014. Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*. 33:131-148.

[11] Roh, K. H. 2010. How to help students conceptualize the rigorous definition of the limit of a sequence. *PRIMUS*. 20(6): 473-487.

[12] Roh, K. H., and Lee, Y. H. 2011. The Mayan activity: A way of teaching multiple quantifications in logical contexts. *PRIMUS*. 21(8): 685-698.

[13] Sealey, V. 2014. A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*. 33: 230-245.

[14] Sealey, V., Deshler, J. M., and Hazen, K. 2014. Strengthening student understanding of mathematical language through verbal and written representations of the Intermediate Value Theorem. *PRIMUS*. 24(2): 175-190.

[15] Selden, J., and Selden, A. 1995. Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*. 29(2): 123-151.

[16] Sellers, M. E., Roh, K. H., and David, E. J. 2017. A comparison of Calculus, transition to proof, and advanced Calculus student quantifications in complex mathematical statements. In Weinberg, A., C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*. (pp. 285-297). San Diego, CA: RUME.

[17] Stewart, J. 2012. *Calculus: Early transcendentals* (7th ed.). Stamford, CT: Brooks/Cole Cengage Learning.

[18] Tall, D., and Vinner, S. 1981. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*. 12(2): 151-169.

[19] Thompson, P. W. 1994. Images of rate and operational understanding of the Fundamental theorem of Calculus. *Educational Studies in Mathematics* 26(2-3), 229-274.

[20] Thompson, P. W., Byerley, C., and Hatfield, N. 2013. A conceptual approach to calculus made possible by technology. *Computers in the Schools*, 30, 124-147.

[21] Williams, S. R. 1991. Models of limit held by college calculus students. *Journal for Research in Mathematics Education*. 22: 219-236.

[22] Zandieh, M. 2000. A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*. 8: 103-127.