Educ Stud Math (2008) 69:217–233
DOI 10.1007/s10649-008-9128-2

Students' images and their understanding of definitions of the limit of a sequence

Kyeong Hah Roh

Published online: 19 July 2008

Abstract There are many studies on the role of images in understanding the concept of limit. However, relatively few studies have been conducted on how students' understanding of the rigorous definition of limit is influenced by the images of limit that the students have constructed through their previous learning. This study explored how calculus students' images of the limit of a sequence influence their understanding of definitions of the limit of a sequence. In a series of task-based interviews, students evaluated the propriety of statements describing the convergence of sequences through a specially designed hands-on activity, called the ε–strip activity. This paper illustrates how these students' understanding of definitions of the limit of a sequence was influenced by their images of limits as asymptotes, cluster points, or true limit points. The implications of this study for teaching and learning the concept of limit, as well as on research in mathematics education, are also discussed.

Keywords Asymptotes · Cluster points · Images · Limits · Rigorous definitions · Sequences

1 Introduction

The concept of limit is one of the most fundamental ideas not only in understanding calculus but also in developing mathematical thinking beyond calculus and in pursuing mathematical rigor (Ferrini-Mundy and Lauten 1993; Tall 1992). Many mathematical concepts depend upon the notion of limit, including the sum of an infinite series, continuity of a function, the derivative, and the integral.

Students' conceptions of limits have long been an important research topic in mathematics education. Relatively early studies (Piaget and Inhelder 1967; Taback 1975) and some more recent ones (Brackett 1991) have explored students' thinking and cognitive development in relation to infinite sequences and limits. The contributions of this type of

K. H. Roh (✉)
Department of Mathematics and Statistics, Arizona State University,
P.O. Box 871804, Tempe, AZ 85287-1804, USA
e-mail: khroh@math.asu.edu

study include designing curricular sequences and deciding specific grade levels to teach the concept of limit.

Research reveals that a large proportion of students have misconceptions about limits. Some misconceptions occur when students do not conceive of infinite processes but just apply the finite processes to solve problems of limits. Those students who do not understand infinite processes confuse the limit with the value of the function or a sequence, or an approximation of the limit (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic 1996). Another misconception about limits arises from regarding the infinite process itself as the limit rather than seeing the limit as the *result* of the infinite process (Vinner 1991). These misconceptions are unavoidable (Davis and Vinner 1986) and are difficult to remediate (Williams 1991). Misconceptions about limits not only affect the understanding of limit itself, but also cause difficulties in subsequent topics such as continuity and differentiability of functions (Bezuidenhout 2001; Cornu 1991) and infinite series (Sierpińska 1987).

The problems in learning the concept of limit become aggravated when the following rigorous definition of convergence of a sequence, called the $\varepsilon - N$ definition in this paper, is introduced to students:

> We call a sequence $\{a_n\}$ to be convergent to a real number A if for any positive number ε, there is a natural number N such that $|a_n - A| < \varepsilon$ for all $n \geq N$ (Apostol 1974, p. 70).

Szydlik (2000) and Williams (1991) found that most students did not adopt the formal view of limits. Cottrill et al. (1996) reported that only a few students understood the formal definition of limit, and that no students in their study applied the formal definition of limit to specific situations.

In line with this observation, this study explored the following research question: How is students' understanding of the definition of the limit of a sequence influenced by the types of images of limits acquired in their previous learning? To answer this question, this study examined the ways that college calculus students understand definitions of the limit of a sequence in relation to the compatibility of their images of limits acquired previously.

2 Images and the development of understanding of the concept of limit

2.1 Dynamic images as sources of students' conceptions of limit

One source of students' conceptions about limits lies in the images expressing the convergence of a sequence. For instance, when students start to study the limit of a sequence, they are usually taught that the limit of a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers is a certain number L which, as n becomes larger, a_n is "approaching" or "getting close to" (Neuhauser 2004; Stewart 2003). Students' mental image of the dynamic motion implied by limits is typically promoted in students' earlier learning experiences, prior to a more formal instructional approach that includes the rigorous definition of limit (Hitt and Lara-Chavez 1999). Such a dynamic image of limits becomes a concept image that students continue to use even after they learn the rigorous definition of limit (Przenioslo 2004).

However, the words used in expressing the dynamic imagery of limits, such as "approaching" or "getting close to," do not precisely convey the mathematical meaning of the concept of limit (Oehrtman 2002; Tall and Vinner 1981; Williams 1991). These expressions instead convey an everyday sense of such words, without the precision of the mathematically rigorous meaning of limits.

2.2 Dynamic images and the rigorous definition of limit

Many attempts have been made to develop instructional methods that aid in the visualization of the rigorous definition; thereby helping students to determine limits accurately as well as to use the rigorous definition of limit properly (Cottrill et al. 1996; Kidron and Zehavi 2002; Mamona-Downs 2001). There have also been various conjectures of dynamic images as to the causes of the difficulties students face in learning the rigorous definitions of limit. For instance, Tall and Vinner (1981) and Williams (1991) conjecture that once students have absorbed the dynamic image of limits from their previous learning, this image may interfere with understanding the rigorous definition of limit. On the other hand, Cottrill et al. (1996) propose that students must develop their ability to grasp the dynamic image of limits in order to precisely understand the rigorous definition of limit. Moreover, Kidron and Zehavi (2002) assert the effectiveness of dynamic images in learning the concept of limit, and suggest a computer animation as a powerful visual interpretation of the dynamic image of limits. Pinto and Tall's (2002) study also revealed that visual images play a positive role in teaching and learning real analysis. Similarly, Navarro and Carreras (2006) showed that a visual instructional method would not only help students build a relevant concept image of limits, but it would also help students move toward a mathematically more rigorous definition. On the other hand, Alcock and Simpson (2004, 2005) could not confirm that learning by visualization was helpful in advanced mathematical thinking.

Examining results from previous studies about students' conceptions of limit shows that it is necessary to explore thoroughly the relationship between students' images of limits and their understanding of the $\varepsilon - N$ definition of limit. However, preceding such an analysis, it should be taken into account that most students who study the $\varepsilon - N$ definition of limit have already constructed their own understanding of limits. Furthermore, these students acquired an image of the concept of limit in their own ways. It would thus be possible for a student to be influenced in understanding the $\varepsilon - N$ definition of limit according to how compatible his/her image of limit is with the $\varepsilon - N$ definition of limit. That is, the propriety of the image of the limit acquired through previous learning can assist or hinder learning the $\varepsilon - N$ definition of limit.

3 Research methodology

This study was conducted in a Midwestern public university in the USA in 2004. Instructors in the mathematics department who were going to teach calculus courses were asked to allow interviews with their students. Four instructors volunteered to participate in this research. The courses they taught were the first quarter calculus for biology majors, the third quarter calculus for engineering majors, and the fourth quarter calculus for engineering majors. Prerequisites for these courses were algebra and pre-calculus. In turn, these calculus courses were prerequisites for real analysis courses which deal with the rigorous definition of limit. Therefore, the students who participated in this study had already encountered limits by means of reading the limit symbol but not by the rigorous definition.

3.1 The survey and the selection of interviewees

The survey was designed to identify characteristics of the target students as well as to select interviewees. The survey consisted of structured questions in which all participants received the same set of questions in the same order. The following checklist provided the specific criteria for selecting participants in this study:

- Those who were acquainted with and had used the limit symbol, but had yet had no experience with any mathematically rigorous process of proofs using the $\varepsilon - N$ definition of limit;
- Those who could determine the value of a specific term of a sequence for the given index, and conversely, determine indices of the terms corresponding to the given value;
- Those who could read and plot graphs; and
- Those who could express their ideas clearly and informatively.

Twenty-one (21) students who enrolled in calculus courses for engineering majors and 12 students who enrolled in a calculus course for biology majors voluntarily took the survey. Among these 33 students, 12 students were selected using the above criteria for task-based semi-structured interviews, and 11 of them completed the whole series of interviews. Pseudonyms are used in this paper, whereby biology students are all given first names beginning with a “B” and engineering students, first names beginning with an “E.”

3.2 Task-based interviews

A series of 1-hour, semi-structured task-based interviews was conducted once a week for 5 weeks. The following 5 types of sequences were covered for the interviews: monotone bounded, unbounded, constant, oscillating convergent, and oscillating divergent sequences. Some examples of the sequences used in the interviews are found in Table 1. During the interviews, students were asked to represent these sequences numerically as well as graphically. Also, for each of these representations of a sequence, they were asked to determine convergence or divergence of the sequence and the limit of the sequence if it should exist.

3.2.1 The ε–strip activity

In addition to the numerical and graphical representations of a sequence, students were also asked to use the tools called “ε–strips” which were specially developed for this study. These ε–strips are made of translucent paper so that students can observe the graph of a sequence through the ε–strips. In addition, each ε–strip is a strip with constant width, and a red line is drawn in its center so as to mark a possible limit value for a sequence on the graph. In the case when the center line of an ε–strip is located at a real number L, the terms of a sequence $\{a_n\}_{n=1}^{\infty}$ inside the ε–strip are in fact those terms satisfying the strict inequality $|a_n - L| < \varepsilon$. Figure 1 illustrates an ε–strip centered on the x-axis and lying on the top of the graph of a monotone bounded sequence $a_n = 1/n$ for any positive integer n.

Students were asked to put ε–strips of different widths on the same graph of the sequence to cover a possible limit point. Then they were asked to observe points on the graph of the sequence distributed inside and outside the ε–strips. In particular, they were asked to determine the number of points outside a given ε–strip; then they were asked to

Table 1 Sequences used in the interviews and students' responses to limits of sequences

Sequences	Imagining limits as asymptotes	Imagining limits as cluster points	Imagining true limit points
$a_n = n/n$	*No limit*	1	1
$a_n = 0$	*No limit*	0	0
$a_n = \begin{cases} 1/n & \text{if } n \leq 10, \\ 1/10 & \text{if } n > 10. \end{cases}$	*No limit*	1/10	1/10
$a_n = 1/n$	0	0	0
$a_n = 1/(2n+1)$	0	0	0
$a_n = (1/2)^{n-1}$	0	0	0
$a_n = n/(n+1)$	1	1	1
$a_n = n$	No limit	No limit	No limit
$a_n = \sqrt{n}$	No limit	No limit	No limit
$a_n = n^2/(n+1)$	No limit	No limit	No limit
$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$	*No limit*	0	0
$a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$	*No limit*	1	1
$a_n = \frac{(-1)^n}{n}$	*0 or no limit*	0	0
$a_n = 1 + \frac{(-1)^n}{n}$	*1 or no limit*	1	1
$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$	*No limit*	*1 and 0*	No limit
$a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$	*0*	*1 and 0*	No limit
$a_n = (-1)^n(1 + 1/n)$	*1 and −1, or no limit*	*1 and −1*	No limit
$a_n = (-1)^n + \frac{1}{n}$	*1 and −1, or no limit*	*1 and −1*	No limit

Italicized cells indicate incorrect answers or incorrect reasoning; un-italicized cells indicate correct answers as well as correct reasoning.

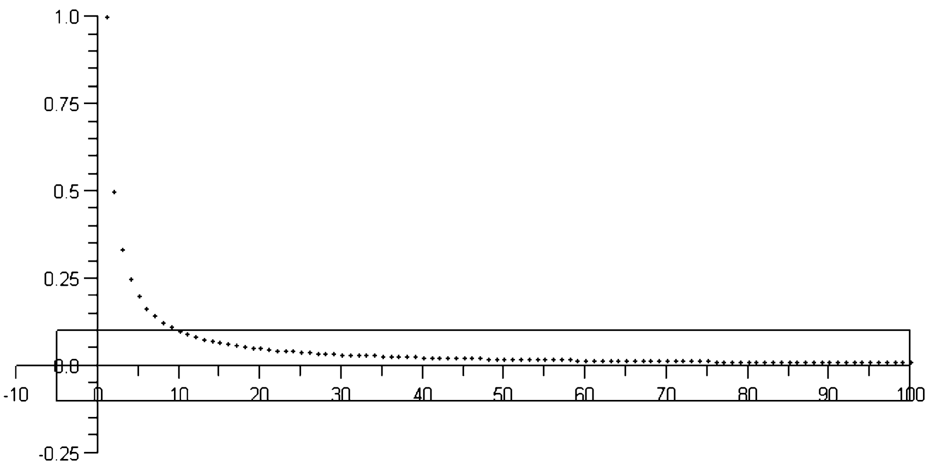

Fig. 1 A graph of the sequence $\{1/n\}_{n=1}^{\infty}$ with an ε–strip

determine the number of points inside the same ε–strip. Throughout this activity, students had an opportunity to visually study the relationship between ε and N in the ε–N definition of the limit of a sequence.

After providing enough opportunities to work with the graphs of sequences and the ε–strips, two "ε–strip definitions" were introduced to students as follows:

– ε–strip definition A: A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as long as the epsilon strip covers L.
– ε–strip definition B: A certain value L is a limit of a sequence when only finitely many points on the graph of the sequence are *not* covered by any epsilon strip as long as the epsilon strip covers L.

In order not to imply the validity of either ε–strip definition, these statements were presented as hypothetical calculus students' ideas. In fact, the participants were comfortable criticizing and revising these ε–strip definitions because the influence of the interviewer's authority and the implication of the term *definition* were minimized by introducing these statements as student definitions.

Students were also asked to compare their answers for the limit of a sequence derived from their own conceptions of limits to the answer obtained by applying ε–strip definition A and ε–strip definition B. Through this procedure, it was hoped that students could determine which ε–strip definitions was a proper definition of the limit of a sequence.

The visualization used in the ε–strip activity is similar to a "microscopic view" or a "zooming-in" of activities described in other studies (Kawski 1997; Mamona-Downs 2001; Navarro and Carreras 2006; Stroyan 1998; Taback 1975). However, the main purpose of the ε–strip activity in this study, which is distinct from the other studies mentioned above, was to investigate how students' understanding of the ε–strip definitions are associated with their images of the limit of a sequence.

In the $\varepsilon - N$ definition of limit, for given $\varepsilon > 0$, an index N must be chosen depending on ε. Then it is determined whether or not terms corresponding to all indices after the chosen N are located within the given error bound ε of L. This is logically equivalent to the statement that only terms corresponding to indices before N can be located outside the error bound ε. That is, for arbitrarily given $\varepsilon > 0$, only finitely many terms exist outside the given error bound ε. Consequently, ε–strip definition B is logically equivalent to the $\varepsilon - N$ definition of limits. Therefore, the reasoning needed to understand the $\varepsilon - N$ definition is also required to understand ε–strip definition B.

4 Results

When the ε–strip definitions were first introduced, most students could not distinguish the difference in meaning between ε–strip definition A and ε–strip definition B. For instance, Elena had thought that the two ε–strip definitions implied each other in the sense that, for infinitely many terms of a sequence to be inside an ε–strip, it must be that only finitely many terms of the sequence could be located outside the ε–strip. The following excerpt reveals her perception:

Interviewer: Those descriptions look different.
Elena: There is a difference. But they kind of imply each other.
Interviewer: You mean student A implies student B, and student B implies student A?

Elena: Yes, because what you are saying, there is infinitely many points on the inside the graph, then it kind of says there have to be finitely many points outside, you know.

This kind of response seemed to occur because students may fail to realize that the complement of an infinite set need not be a finite set. Some students may confirm their reasoning by imagining monotone, constant, or oscillating convergent sequences, for which both ε–strip definitions are equivalent. They may not think about oscillating divergent sequences, which reveal the difference between ε–strip definition A and ε–strip definition B.

It should be noted that the belief that both definitions were equivalent to each other was often temporary, although it seemed an important stage in understanding the ε–strip definitions. Students actually acknowledged the difference between ε–strip definition A and ε–strip definition B after seeing an oscillating divergent sequence $a_n = (-1)^n(1 + 1/n)$ via the ε–strip activity (see Fig. 2).

Indeed, students who regarded both 1 and −1 as limits of the sequence $a_n = (-1)^n(1 + 1/n)$ perceived their early response as false and determined ε–strip definition B not to be appropriate when working with the oscillating divergent sequence $a_n = (-1)^n(1 + 1/n)$. The following excerpt with Emma is such a case.

Emma: $a_n = (-1)^n(1 + 1/n)$

Interviewer: Can you determine whether student A and student B are correct or not by using −1?

Emma: Okay. I mean it's just the same argument: −1 is a limit of the sequence by student A's argument because infinitely many points are covered by the epsilon strip. And by student B's definition, −1 is not a limit of the sequence because there are infinitely many points outside the epsilon strip.

Interviewer: Umm. What do you think?

Emma: So, well, student B is wrong, because it is saying it can't, −1 would not be a limit of the sequence. But it [−1] is [a limit of the sequence].

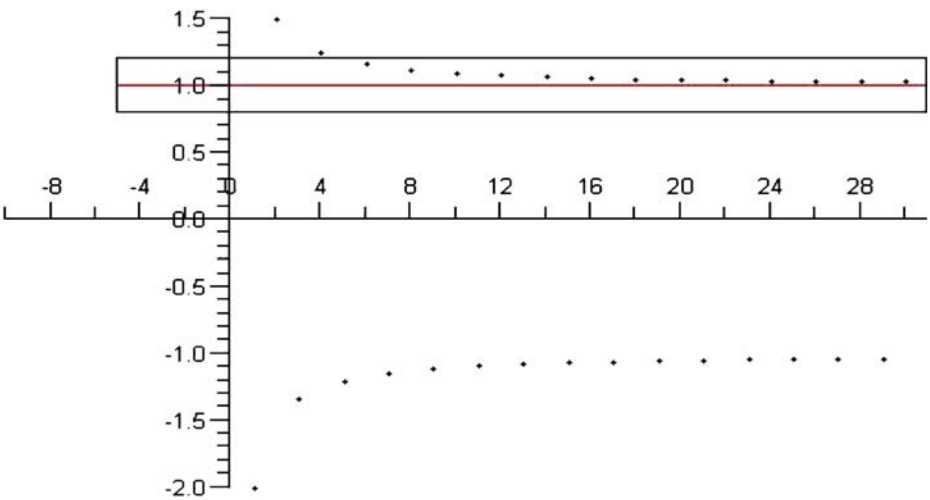

Fig. 2 A graph of the sequence $\{(-1)^n(1 + 1/n)\}_{n=1}^{\infty}$ with an ε–strip

On the other hand, students who decided the sequence $a_n = (-1)^n(1 + 1/n)$ was a divergent sequence also perceived their early response as false. However, unlike Emma, these students determined ε–strip definition A as not appropriate. For instance, Becky realized that the ε–strip definitions did not imply each other and determined only ε–strip definition B to be appropriate as follows:

Becky: $a_n = (-1)^n(1 + 1/n)$

Becky: Well, like I always thought of this example [A], but now I have seen this one [B]. Like, you kind of have to include B, too, because this [A] could be misleading. If you see that there is infinitely many inside, but you have to consider all these outside, too. They are approaching something else. It is different.

Once a student properly understood both ε–strip definitions, his/her response on the appropriateness of the ε–strip definitions became stable and consistent. From that moment, students' responses to the propriety of the ε–strip definitions were classified into three groups with regard to whether or not they accepted each of the ε–strip definitions. The following subsections describe these three groups of student responses to the ε–strip definitions in relation to their images of the limit of a sequence.

4.1 Group I: neither ε–strip definition is correct

Some students responded that neither ε–strip definition was appropriate as the description of a limit of a sequence. For instance, Brian, after graphing a constant sequence, pointed out that all terms of the sequence are overlapped by a straight line, and he believed that the straight line could not be a limit of the sequence.

Brian: $a_n = 1$

Interviewer: Does this sequence have a limit?
Brian: I want to say no, just because it would be a straight line, always the values are gonna be 1.
Interviewer: How can you tell this?
Brian: Because it is not approaching any number.
Interviewer: What do you mean by approaching?
Brian: The value is just always 1, so there is [pause]. Guess my definition of limit is approaching a number but not reaching it.
Interviewer: Okay.
Brian: This [sequence] is not approaching a number, but it is already there. So, and then, it is never changing the value from the value of 1.

Since all terms of a constant sequence are on the center line of any given ε–strip, infinitely many terms of the sequence must be inside and none of them can be outside the ε–strip. Hence, applying both ε–strip definitions A and B to constant sequences, the constant sequences must be convergent. However, this result conflicted with Brian's belief that constant sequences are not convergent. In conclusion, Brian did not accept either of the ε–strip definitions as an appropriate description of the limit of a sequence. Such a determination occurred not only in the case of constant sequences but also in the case of oscillating sequences such as the sequence $\{a_n\}_{n=1}^{\infty}$ defined as 1 for each odd term and 1–1/n for each even term (see Fig. 3).

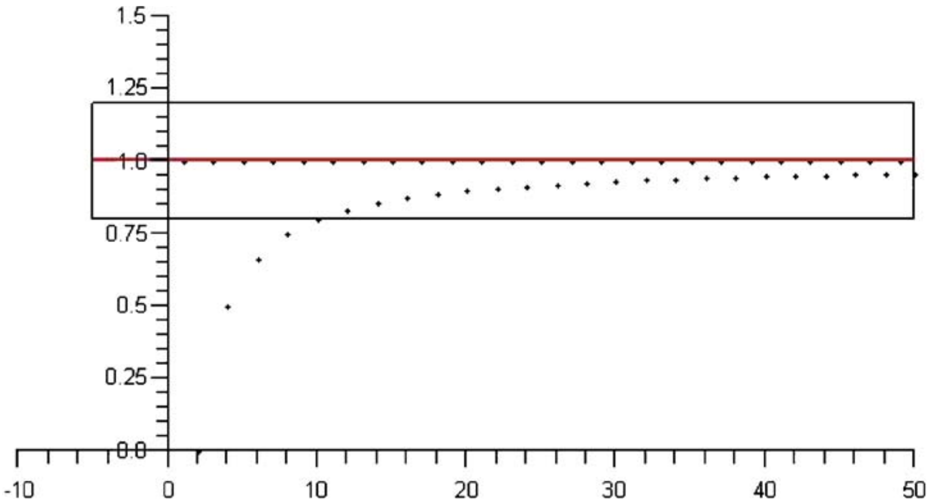

Fig. 3 A graph of the sequence $a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$ with an ε–strip

When Brian tested the appropriateness of the ε–strip definitions on the oscillating sequence, he observed that infinitely many points on the graph of the sequence exist inside a given ε–strip, and only finitely many points exist outside the ε–strip. He then recognized that this case is the same as constant sequences in the sense that odd terms of the sequence are located on the center line of an ε–strip centered at $y = 1$. Therefore, Brian considered both ε–strip definitions A and B to be inappropriate for descriptions of limits of sequences. The following excerpt explains his reasoning:

Brian: $a_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$

Brian: Okay. So, there is n. Okay. So one [a_1] will be on 1. Two [a_2] will be at 1/2. Three [a_3] will be on 1. Four [a_4] will be 3/4. Five [a_5] will be 1. Six [1/6] will be a small fraction. So the epsilon strip covers 1, there is a finite number of points outside of the strip which will work for this [B]. And an infinite number of points are covered by the strip which will work for this [A]. But…

Interviewer: So you mean that 1 is a limit of this sequence?

Brian: Umm. I still disagree that a straight line is a limit of the sequence.

Similarly to Brian, students in Group I agreed that infinitely many terms of the sequence should be inside ε–strips, but insisted that terms of the sequence should not be located on the center line of ε–strips. In line with this viewpoint, they wanted to revise the ε–strip definitions by adding the condition that "no terms of the sequence should be located on the center line of ε–strips."

It was observed that these students tended to consider a sequence convergent if the terms of the sequence approached a certain value, but none of the terms of the sequence was equal to the value. As a consequence of this conception of convergence, these students tended to imagine limits of a sequence as *asymptotes*, that is, lines that the graph of the sequence was getting arbitrarily close to but not surpassing nor intersecting. In this study these students were categorized as those who imagine limits as asymptotes.

In fact, students in the asymptote category could find the *asymptotes* in the case of monotone bounded sequences, for instance, $a_n = 1/n$ ($n \in \mathbb{N}$); hence they could properly determine such sequences to be convergent. However, they failed to properly determine convergence of non-monotone sequences. For instance, as seen above, in the case of the constant sequence $a_n = 1$ ($n \in \mathbb{N}$), Brian pointed out that this sequence is not getting close to any real number, so he determined that constant sequences would not be convergent to any value. Besides the condition "getting close to," students in the asymptote category considered the condition "not equal to" or "not surpassing" as crucial in determining the convergence of sequences. As seen above, in the case of an oscillating sequence, whose odd terms are defined as 1 and even terms are defined in the form of $1 - 1/n$, Brian believed that 1 could be the only potential value for the limit. However, 1 could not be the limit because the odd terms take on the value 1, so the terms of the sequence do not satisfy the condition "not equal to." Some students in the asymptote category also determined for an oscillating convergent sequence $\{(-1)^n/n\}_{n=1}^{\infty}$ not to be convergent because they could not find proper asymptotes in mentioning that the sequence surpasses 0. Other students in this category determined for an oscillating divergent sequence $\{(-1)^n(1+1/n)\}_{n=1}^{\infty}$ to be convergent since it has proper asymptotes. In fact, they mentioned its subsequences with even indices and odd indices are getting close to but are not surpassing 1 and -1, respectively.

4.2 Group II: ε–strip definition A is correct but not B

Some students believed that ε–strip definition A was appropriate as a description of limits but not ε–strip definition B. For instance, Emily asserted that ε–strip definition A describes well the phenomenon that there are always infinitely many terms of the constant sequence inside the ε–strip regardless of the width of the ε–strip. As a consequence, she believed ε–strip definition A to be applicable in determining the limit of a sequence.

Emily: $a_n = 1$

Emily:	This [A], this is, I agree with. What Student A is saying is right, or great.
Interviewer:	How can you tell this?
Emily:	[It is] because a certain number L, which is 1 in this case, is a limit when infinitely many points on the graph of the sequence are covered by any epsilon strips, so as we were saying, even if these [ε–strips] are getting smaller and smaller, a certain number L, which is 1, will still be within epsilon strips. It's the limit.

Students in Group II explicitly argued that the description in ε–strip definition B is inappropriate for the limit of a sequence. For instance, Emma thought that a convergent sequence could have multiple limits, and pointed out that by ε–strip definition A, both 1 and -1 are taken as limits of the sequence $a_n = (-1)^n(1+1/n)$, whereas by ε–strip definition B, it is impossible to have both limits.

Emma: $a_n = (-1)^n(1+1/n)$

Emma:	Okay. Honestly here is a situation that this one [A] does work, and this one [B] doesn't.
Interviewer:	Can you explain it one more time?
Emma:	Umm. A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any epsilon strip as

long as it covers L. And that [A] is true. There are infinitely many points and this is on the sequence. And here this [B] is not true… there are infinitely many points outside the epsilon strip while the epsilon strip is in fact [centered at] a limit of the sequence. So this [B] isn't true here. Because the student [B] is saying that for a value to be a limit of a sequence, finitely many, only finitely many points should not be covered by any epsilon strip. And here, by this argument, 1 would not be a limit of the sequence. So, well, student B is wrong.

Interviewer: But 1 is a limit of the sequence?

Emma: Right.

By closely examining their conception of limits, it was observed that students in Group II tended to consider sequences convergent if their terms approach or are equal to some values. In fact, students in Group II looked to see if the terms of a sequence were clustered around a certain value. To be precise, they examined if infinitely many terms of a given sequence were located close to a certain value, or were within a sufficiently small neighborhood of a certain value they thought was its limit. Such a value is mathematically a "cluster point" defined as follows:

> A point x is called a *cluster point* of the sequence $\{x_n\}$ if for every $\varepsilon > 0$ there are infinitely many values of n with $|x_n - x| < \varepsilon$ (Marsden and Hoffman 2000, p.53).

These students thus imagined limits of sequences as cluster points. It is not easy for students to conceptually distinguish between cluster points and limits of sequences. In general, the limit of a sequence is also a cluster point. But a cluster point of a sequence needs not be a limit of the sequence. For example, the sequence $\{(-1)^n\}_{n=1}^{\infty}$ has two cluster points, 1 and −1, but neither of them is the limit of the sequence.

Students in the cluster-point category properly determined the convergence of sequences in the case of monotone bounded sequences, just as students who imagined limits as asymptotes did. However, the viewpoint that the limit of a sequence *can be equal to* terms of the sequence is definitely different from the viewpoint of limits as asymptotes. The following excerpt illustrates Emma's viewpoint.

Emma: $a_n = \begin{cases} 1/n & \text{if } n \leq 10, \\ 1/10 & \text{if} n > 10. \end{cases}$

Emma: The limit would be 1/10.

Interviewer: The limit would be 1/10? How can you tell this?

Emma: Because to be this first part, $1/n$, is approaching and then reaches 1/10. And then for the second condition, [n] to be greater than 10, a_n equals 1/10. All the terms will be 1/10, so the sequence is getting closer to 1/10 until it reaches it, and it equals 1/10 thereafter.

By applying the criterion of "getting close to or equal to," students in the cluster-point category also determined convergence of oscillating sequences differently from students who imagined limits as asymptotes. For instance, considering the sequence defined by 0 for each odd term and $1/n$ for each even term, Emily determined that 0 is the limit of the sequence by observing that the sequence is approaching 0 or equal to 0.

Emily: $a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}$

Interviewer: Does this sequence have a limit?
Emily: Umm, I want to say yes.
Interviewer: Yes? How can you tell this?
Emily: Well, this one is always 0, I mean, if n is odd, it [a_n] is always 0. And then if it [n] is even, it [a_n] approaches 0. So I assume that limit is 0.

On the other hand, imagining limits as cluster points misleads students into determining that oscillating divergent sequences actually have multiple limits. For instance, given the sequence defined by 1 for each even term and 0 for each odd term, Erica determined both 1 and 0 to be limits of the sequence by observing that both 1 and 0 were cluster points of the sequence.

Erica: $a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

Erica: And I kind of think like maybe 1 and 0 are, are limits. This is 0 here, and this is 1 here.
Interviewer: Okay.
Erica: They kind of are all the limits because no matter what n you plug in, you would have some huge [number of points] around here [1 and 0]. The points always are gonna be on these two lines here. They are never gonna be out there.

Students who imagined limits as cluster points focused on the closeness of points to a certain value as the index of the sequence varies. In fact, the statement of ε–strip definition A focuses on how many points are closely located around a certain value. Therefore, students in the cluster-point category considered the statement of ε–strip definition A as similar to their images of convergent sequences.

4.3 Group III: ε–strip definition B is correct but not A

Finally, there were students who insisted that only ε–strip definition B exactly describes the limit of a sequence, in admitting only finitely many terms of the sequence outside any error bound. The following is an excerpt from Elena, a student in this group.

Elena: $a_n = (-1)^n(1 + 1/n)$

Elena: I think student B is right, …completely right.
Interviewer: In what sense? Can you explain why student B is completely right?
Elena: Because yeah, for any … for a sequence has a limit … that is true that only a few points would be outside the strip, and … all the rest of the points, which would be infinitely many points, would be inside the strip and … and it also says any strip so it doesn't matter how small the strip is, … there is only finite amount [of terms] outside.

On the other hand, Elena observed that infinitely many points of the sequence $a_n = (-1)^n(1 + 1/n)$ are located in any ε–strip centered at either 1 or −1, and by ε–strip definition A, this result would force her to accept both 1 and −1 as limits of the sequence which contradicted her conception of limit. Therefore, she pointed out that ε–strip definition A could not properly describe the limit of the sequence.

Elena: $a_n = (-1)^n(1 + 1/n)$

Elena: Well, now this one [A] keeps, keeps presenting wrong. Because … because L, this [ε–strip] is covering our L, our supposed L, our hypothetical L, and there still, there is an infinite amount of points inside the strip. But there is also an infinite amount of points outside the strip as well. And so even though it [A] says that, it's not taking into account these points [outside of the ε–strip].

In fact, students in Group III checked to see if a given sequence was "getting close to or equal to a *unique* value". To be precise, these students looked for a unique value L so that the difference between each term of the sequence and L was decreasing to 0 or equal to 0. Such a conception is compatible with the mathematical concept of the limit of a sequence. These students were thus classified as those who imagine limits as true limit points.

Students in the limit-point category are definitely different from those of the two other categories, asymptotes or cluster points. First, these students include the possibility of terms of the sequence "being equal to" the limit, which students in the asymptotes category do not. Second, students in the limit-point category accept the uniqueness of the limit which students in the other categories do not think is necessary. In the following examples, the difference between the limit-point category and the asymptote and cluster-point categories is discussed in detail.

Indeed, students in the limit-point category point out that "approaching" or "getting close to but not equal to" is inappropriate for describing the convergence of sequences. For instance, Elena suggested examples such as a constant sequence $a_n = 0$ for any natural number n or an oscillating sequence defined by 0 for each odd term and $1/n$ for each even term, neither of which is consistently "approaching" any value but both of which are convergent.

Elena: $a_n = 0$

Interviewer: Does this sequence have a limit?
Elena: Yes, it has a limit. That will be 0.
Interviewer: How can you tell this?
Elena: That's because every, every, every a_{n} is gonna equal 0. I think you can still have the limit of a constant equal to the constant.
Interviewer: Does this sequence approach 0?
Elena: I don't … I don't … I think approaching may be the wrong term for this. Because there is that number, you know? Approaching indicates that, approaching indicates that… the sequence, …gets … infinitely close to the number, but never quite hits it. You know what I am saying? … You have a couple of examples of that. But … this [$a_{\mathrm{n}} = 0$], on the other hand, this is a completely different story because it is that number [0] for the whole time. … This one, $\left[a_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1/n & \text{if } n \text{ is even.} \end{cases}\right]$ half the time that is that number [0].

Since students in the limit-point category also examine if a sequence is "getting close to or equal to" a certain value, one might consider the conception of these students to be similar to that of students in the cluster-point category. However, the acceptance of the *uniqueness* of the limit is a crucial factor in distinguishing students in the limit-point category from those in the cluster-point category. Students in the limit-point category did not consider a sequence convergent even if each of its subsequences was getting close to or equal to a certain value unless all subsequences were approaching the same value. Therefore, the uniqueness of the limit of a sequence constitutes a meaningful part of the limit-point students' criterion for convergence of sequences, whereas students in the cluster-point category do not worry about uniqueness. Becky's description of the limit of a sequence reveals her understanding of the uniqueness of the limit of a sequence:

Interviewer: What is your criterion in deciding whether it is a limit or not?

Becky: There has to be like, as x approaches infinity, all the values of the sequence have to be approaching that number, and like all the points. They can't be like some of them, half approaching like 1 and half of them approaching -1. They all have to be approaching the same number. It does not matter if there were positive or negative as long as it is approaching that number.

It should also be noted that students in the limit-point category use the expression "approaching" with a broader, less literal meaning than students in the asymptote category. In fact, they admit a value as the limit of a sequence even if terms of the sequence surpass the limit value. However, to students who imagine limits as asymptotes, "approaching" means getting close to but not equal to and not surpassing. In contrast to students in the asymptote category, students in the limit-point category think that it is not important whether or not the sequence is surpassing a value anticipated as its limit in determining the limit of a sequence. Table 2 summarizes the main characteristics of the three images of limit and the key difference among them.

5 Discussion

As mentioned in Section 2.2 of this paper, the relationship between students' (mis) conceptions of limits and their image of dynamic motion in understanding limits has been discussed and studied by Cottrill et al. (1996), Kidron and Zehavi (2002), Pinto and Tall (2002), Tall and Vinner (1981), Williams (1991), and others. The study reported here may add some new perspectives to the conversation.

5.1 Connection to other literature on students' conceptions and images of limits

In the case of students imagining limits as asymptotes, they seem to conceptualize limits via the dynamic images of asymptotes. Such a conception gives rise to errors in determining

Table 2 Students' images of limit and their responses to appropriateness of ε–strip definitions

Images of limit	Asymptotes	Cluster points	Limit points
Criteria for convergence	Getting close to but not equal to	Getting close to or equal to	Getting close to or equal to a unique value
ε–strip definition A is proper	No	Yes	No
ε–strip definition B is proper	No	No	Yes

the convergence of a sequence if its graph does not match up with the image of an asymptote. The image of cluster points as limits leads to an error similar to that of asymptotes. As mentioned above, these incomplete dynamic images hinder students in properly understanding the definition of limit, which is in line with the findings of Tall and Vinner (1981), Williams (1991), and others.

In the case of students in the limit-point category, they still seem to conceptualize limits via the imagination of dynamic motion. Nonetheless, their images of limits were compatible with the rigorous definition of limits. As a consequence, their images are conducive to understanding the rigorous definition of limit properly. This finding supports the work of Cottrill et al. (1996).

It has been an important issue whether or not it is desirable to teach the concept of limit based on the dynamic images of limits. In point of fact, most students encounter the intuitive, dynamic approach to limits before they encounter the formal definition. The present study suggests that the real issue is not whether to use dynamic images in instruction, but rather, how to induce dynamic images that are compatible with the definition of limit.

5.2 Implications for teaching and learning the definition of limit

The present study shows that students' understanding of definitions of limits is closely related to whether or not their previously constructed images of limits are compatible with the mathematical concept of limit. Other questions related to teaching and learning of the definition of limit include: How are images that are incompatible with the $\varepsilon - N$ definition constructed through mathematical practice? Are there any instructional innovations that could modify such images into a proper image of limits that could help students understand and learn the $\varepsilon - N$ definition?

Introductory calculus textbooks suggest the expressions "approaching" or "getting close to" as an instructional idea to help students easily understand the $\varepsilon - N$ definition (Neuhauser 2004; Stewart 2003). However, it is questionable whether or not students in fact properly understand the meaning of these phrases in a mathematical context. If students are not guided to recognize how the meaning of these expressions in mathematics differs from that in everyday contexts, it is possible that students will construct images incompatible with the $\varepsilon - N$ definition.

According to the present study, what students mean by a sequence "approaching" a value varied according to their images of the limit of a sequence. One student Brian said "Guess my definition of limit is approaching a number but not reaching it." This is yet another instance of the classic observation that a limit is something one can "approach but never reach." However, Brian's interpretation of "approaching" is appropriate for describing an asymptote. Brian admitted that constant sequences would satisfy ε–strip definitions. Nonetheless, he responded that constant sequences would not have limits, according to his interpretation of limits as asymptotes. Thus, Brian did not accept the ε–strip definitions as proper descriptions for the limit of a sequence. That is, students could imagine an asymptote for the limit of a sequence based on their understanding so far. Their image of limits as asymptotes in turn influenced their understanding of the definition of a limit. This result implies that the teaching and learning of the definition of limit need to help students build on their conceptual images at the same time that they elaborate the meaning of everyday expressions to the precision required in a mathematical context.

In most contexts where the concept of the limit of a sequence is introduced, it is likely that monotone sequences such as $\{1/n\}_{n=1}^{\infty}$ are generally used for referent contexts

(Neuhauser 2004; Stewart 2003). There is no doubt that investigating the limit of a monotone sequence plays an important role in understanding the limit of a sequence. However, if students have only monotone sequences in mind, it may hinder them from distinguishing the concept of limits from the concept of cluster points. When an oscillating sequence $\{(-1)^n(1+1/n)\}_{n=1}^{\infty}$ was given, one student, Erica, responded that "[1 and −1] are all the limits because no matter what n you plug in, you would have some huge [number of points] around here…. I can see that there are almost like two separate things based on how their graph looks." Erica seemed to check where terms of the sequence were clustered around, and envisioned two different monotone subsequences $\{1+1/(2k)\}_{k=1}^{\infty}$ and $\{-1-1/(2k+1)\}_{k=1}^{\infty}$ each of which converged to 1 and −1, respectively. Her focus on monotone sequences allowed her to accept multiple limits for this oscillating sequence. In fact, her mental image based on monotone sequences seems to be compatible, not with the $\varepsilon - N$ definition of limit, but with the definition of a cluster point. Thus it is important that the examples of sequences used in helping students conceptualize the limit of a sequence comprise more than just monotone sequences.

It is known that improper concept images are hard to convert into a correct image once an image is coherently established in students' minds (Williams 1991). In relation to this issue, Brousseau (1997), Cornu (1991), and Fischbein (1987) have proposed the necessity of creating a learning environment in which students are made aware of difficulties and given the opportunity to reflect on their own ideas and epistemological obstacles. Concerning instructional treatment to help students develop images of limits compatible with the $\varepsilon - N$ definition, it is worth noting that the ε–strip activity carried out as a research tool in this study may be effectively used in classroom. Students in this study often expressed cognitive dissonance when faced with constant sequences or oscillating divergent sequences. It is likely that these students experienced such cognitive dissonance due to the discrepancy between their images of limits and what they understood from the ε–strip definitions. But more interestingly, some of these students actually resolved their cognitive dissonance through the ε–strip activity, in particular, by comparing their conceptions of limits with the ε–strip definitions (Roh, 2005). Therefore, the ε–strip activity could let students encounter epistemological obstacles, recognize the subtle differences between the concept of limit and other concepts such asymptotes or cluster points, and eventually overcome the epistemological obstacles in learning the $\varepsilon - N$ definition.

Acknowledgement The author thanks Professor Sigrid Wagner, Professor Alfinio Flores, and three anonymous reviewers for their helpful comments on earlier versions of the article.

References

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *57*, 1–32.

Alcock, L., & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, *58*, 77–100.

Apostol, T. (1974). *Mathematical analysis*. Reading, MA: Addison-Wesley.

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*, 487–500.

Brackett, J. D. (1991). *The association of mathematical context with students' responses to tasks involving infinity*. Dissertation, University of Georgia.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.) Dordrecht, The Netherlands: Kluwer Academic.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Dordrecht, The Netherlands: Kluwer Academic.
Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*, *15*, 167–192.
Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*, 281–303.
Ferrini-Mundy, J., & Lauten, D. (1993). Teaching and learning calculus. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 155–176). New York: Macmillan.
Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: D. Reidel.
Hitt, F., & Lara-Chavez, H. (1999). Limits, continuity and discontinuity of functions from two points of view: That of the teacher and that of the student. In L. Bills (Ed.), *Proceeding of the British Society for Research into Learning Mathematics*, *19*, 49–54.
Kawski, M. (1997). How CAS and visualization lead to a complete rethinking of an introduction to vector calculus. In W. Fraunholz (Ed.), *Proceeding of the Third International Conference on Technology in Mathematics Teaching*. Koblenz, Germany.
Kidron, I., & Zehavi, N. (2002). The role of animation in teaching the limit concept. *The International Journal of Computer Algebra in Mathematics Education*, *9*, 205–227.
Mamona-Downs, J. (2001). Letting the intuitive bear on the formal: A didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics*, *48*, 259–288.
Marsden, J. E., & Hoffman, M. J. (2000). *Elementary classical analysis* (2nd ed.). New York: Freeman.
Navarro, M., & Carreras, P. (2006). Constructing a concept image of convergence of sequences in the van Hiele framework. *Research in Collegiate Mathematics Education*, *VI*, 61–98.
Neuhauser, C. (2004). *Calculus for biology and medicine* (2nd ed.). Upper Saddle River, New Jersey: Pearson Education.
Oehrtman, M. (2002). *Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning*. Dissertation, The University of Texas.
Piaget, J., & Inhelder, B. (1967). *The child's conception of space* (F. J. Langdon & J. L. Lunzer., Trans.). New York: The Norton Library.
Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics*, *22*, 2–10.
Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics*, *55*, 103–132.
Roh, K. (2005). *College students' intuitive understanding of the limit of a sequence and their levels of reverse thinking*. Dissertation, The Ohio State University.
Sierpińska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, *18*, 371–397.
Stewart, J. (2003). *Calculus: Early transcendental single variable* (5th ed.). Belmont, CA: Brooks/Cole-Thomas Learning.
Stroyan, K. D. (1998). *Mathematical background: Foundations of infinitesimal calculus*. Retrieved from http://www.math.uiowa.edu/~stroyan/research.htm. (June 8, 2007)
Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of function. *Journal for Research in Mathematics Education*, *31*, 256–276.
Taback, S. (1975). The child's concept of limit. In M. F. Rosskopf (Ed.), *Children's mathematical concepts* (pp. 111–144). New York: Teachers College Press.
Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495–511). New York: Macmillan.
Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht, The Netherlands: Kluwer Academic.
Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, *22*, 219–236.