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SUBJECT-MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE: PROSPECTIVE SECONDARY TEACHERS AND THE FUNCTION CONCEPT

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This article investigates teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge in the context of teaching the concept of function. During the first phase of data collection, 152 prospective secondary teachers completed an open-ended questionnaire concerning their knowledge about function. In the second phase, an additional 10 prospective teachers were interviewed after responding to the questionnaire. The analysis shows that many of the subjects did not have a modern conception of function. Appreciation of the arbitrary nature of functions was missing, and very few could explain the importance and origin of the univalence requirement. This limited conception of function influenced the subjects' pedagogical thinking. Therefore, when describing functions for students, many used their limited concept image and tended not to employ modern terms. In addition, many chose to provide students with a rule to be followed without concern for understanding.

Teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge are still very much unknown. This is due, in part, to a change in conceptions of teachers' subject-matter knowledge that has taken place throughout the years. Not many years ago, teacher subject-matter knowledge was defined in quantitative terms—by the number of courses taken in college or teachers' scores on standardized tests (Ball, 1991; Wilson, Shulman, & Richert, 1987). But these "measures" are problematic, since they do not represent teachers' knowledge of the subject matter. In recent years, teachers' subject-matter knowledge has been analyzed and approached more qualitatively, emphasizing knowledge and understanding of facts, concepts, and principles and the ways in which they are organized, as well as knowledge about the discipline; that is, ways to establish truth (Ball, 1988, 1991; Even, 1990; Kennedy, 1990; Leinhardt & Smith, 1985; Shulman, 1986; Tamir, 1987; Wilson et al., 1987).

Another category of subject-matter-specific knowledge of teachers that has gained greater attention in recent years is pedagogical content knowledge. This kind of knowledge is described as knowing the ways of representing and formulating the subject matter that make it comprehensible

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to others as well as understanding what makes the learning of specific topics easy or difficult (Ball, 1988; Even & Markovits, in press; Lampert, 1986; Shulman, 1986, 1987; Tamir, 1987; Wilson et al., 1987).

Even though it is usually assumed that teachers' subject-matter knowledge and pedagogical content knowledge are interrelated (Ball, 1991; Buchmann, 1984; Kennedy, 1990; Shulman, 1986, 1987; Wilson et al., 1987), there is little research evidence to support and illustrate the relationships. The study reported in this article investigates this issue in the context of teaching the concept of function. It draws on a theoretical framework of subject-matter knowledge for teaching mathematical concepts in general and the function concept in particular (Even, 1990). The framework is based on a qualitative analysis of teachers' subject-matter knowledge of mathematics. It consists of the following seven aspects, the first of which is discussed in this paper: (a) essential features—what is a function?; (b) different representations of functions; (c) alternative ways of approaching functions; (d) the strength of the concept—the inverse function and the composition of functions; (e) basic repertoire—functions of the high-school curriculum; (f) different kinds of knowledge and understanding of the function concept; and (g) knowledge about mathematics. The building of the framework is based on integrated knowledge from several bodies of work: the role and importance of the function in the discipline of mathematics and in the mathematics curriculum; research and theoretical work on the learning, knowledge, and understanding of functions in particular and other mathematical concepts in general; and research and theoretical work on teachers' subject-matter knowledge and its role in teaching.

This article focuses on one aspect of subject-matter knowledge for teaching functions, an aspect that concentrates on the essential features of the modern concept of function. More specifically, this study addresses the question, What are the essential features of the modern concept of function? This concept has undergone an interesting evolution (Bennett, 1956; Bottazzini, 1986; Eves, 1983; Freudenthal, 1983; Hamley, 1934; Hight, 1968; Kleiner, 1989; Kline, 1972; Malik, 1980). Developments in mathematics have changed the concept of function from a curve described by a motion (17th century) to an analytic expression made up of variables and constants representing the relation between two variables with its graph having no "sharp corners" (18th century). Then, new discoveries and rigorization led to the modern conception of a function as a univalent correspondence between two sets. More formally, a function f from A to B is defined as any subset of the Cartesian product of A and B, such that for every $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

The evolution of the function concept is sometimes described as a move from a dynamic-dependency notion to a static-set-theoretic one (Freudenthal, 1983), or from an operational notion as a process to a structural notion as an object (Sfard, 1991). (For an extended discussion of this issue, see Kieran, in press.) In any case, as Freudenthal points out, two essential features of the modern concept of function have evolved: arbitrariness and univalence.

The *arbitrary nature* of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the relationship means that functions do not have to exhibit some regularity, be described by any specific expression or particular shaped graph. The arbitrary nature of the two sets means that functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers. Whereas the arbitrary nature of functions is implicit in the definition of a function, the *univalence* requirement, that for each element in the domain there be only one element (image) in the range, is stated explicitly. As the history of the development of the concept of function shows, univalence was not required at the beginning. Freudenthal (1983) attributes this requirement to the desire of mathematicians to keep things manageable. The development of advanced analysis created the need to deal with differentials of orders higher than one and, therefore, to distinguish independent from dependent variables. In such a case, it became too difficult to work with multivalued symbols, and the univalence requirement was added to the definition of a function.

The 18th and 19th century mathematicians struggled with the idea of the arbitrariness of functions. Interestingly, so do today's students, even though the concept of function is defined in today's textbooks in a modern sense. Several studies (Dreyfus & Eisenberg, 1983, 1987; Lovell, 1971; Markovits, Eylon, & Bruckheimer, 1983, 1986; Marnyanskii, 1975; Thomas, 1975; Vinner, 1983; Vinner & Dreyfus, 1989) found that many students do not hold a modern conception of function (see also a discussion by Kieran, in press). For example, many students appear to hold a linear prototypic image of functions. Many expect graphs of functions to be "reasonable" and functions to be representable by a formula. Students often do not include as functions constant functions, split-domain and piecewise functions, or functions obtained by composition. In general, the findings point to a limited view of functions, caused by students having some specific expectations about functions and their behavior. Surprisingly, Vinner and Dreyfus (Vinner, 1983; Vinner & Dreyfus, 1989) found that this was the case even when students were able to provide a correct modern definition. In other words, defining a function in modern terms did not necessarily reflect a modern concept image[1] of a function. The inconsistent behavior described by Vinner and Dreyfus is a specific case of the compartmentalization phenomenon described in Vinner, Hershkowitz, and Bruckheimer (1981). This phenomenon occurs when a person has conflicting schemes in his or her cognitive structure. Different situations stimulate different schemes, and the person appears to behave either inconsistently or in an irrelevant way.

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[1]The term "concept image" is taken from Vinner (Vinner, 1983; Vinner & Dreyfus, 1989), who defines concept image as the mental picture of the concept (i.e., the set of all "pictures" that have ever been associated with the concept in the person's mind) together with the set of properties associated with the concept (in the person's mind). This concept image can clearly differ from person to person.

Since the process of learning is influenced by the teacher, it is therefore important to understand how teachers explain what a function is to students, what they emphasize and what they do not; and what ways they choose to help students understand. Teachers' pedagogical content knowledge is influenced by their subject-matter knowledge. However, the interrelations between the two are still very much unknown. This study investigates the interrelations between teachers' content knowledge and pedagogical content knowledge related to two essential features of the concept of function, arbitrariness, and univalence. The study reported in this paper is part of a larger study of different aspects of prospective secondary mathematics teachers' knowledge and understanding about functions (Even, 1989).

METHODOLOGY

Subjects

General Background

The participants in this study were 162 prospective secondary mathematics teachers in the last stage of their formal preservice preparation. The subjects came from eight midwestern universities in the United States. Five are large universities with extensive teacher education programs; the other three are state universities, previously teacher colleges, with an emphasis on teacher education. One hundred and fifty-two subjects from seven of the universities participated in the first phase of the study, in which they were administered a questionnaire. Ten subjects from two universities participated in the second phase, in which they completed the same questionnaire and were interviewed in depth. The first-phase subjects were students enrolled in mathematics methods classes (i.e., mathematics pedagogy classes) in which the instructors were willing to devote one hour of class time to administering questionnaires to their students. All students who were present on the day of questionnaire administration were included. Since the development of the interview for the second phase was based, in part, on an analysis of the first-phase data, second-phase subjects were not chosen from among the first-phase subjects. However, an attempt was made to select second-phase subjects who were similar to first-phase subjects with respect to sex, age, academic background, and grades. In general, most subjects were in their early 20's, and were equally distributed by gender.

Academic Background

The subjects had completed most of the required mathematics coursework. About one-third of the first-phase subjects and two-thirds of the second-phase had taken an advanced calculus course. All first-phase subjects and three second-phase subjects were enrolled in the required mathematics methods course during this study; seven subjects from the second phase had already completed it. More than two-thirds of the subjects had had some field experience, usually pre–student teaching in the first phase and student teaching in the second. Thus, an average second-phase subject had somewhat stronger mathematics and teaching background than an average first-phase subject, having taken an advanced calculus course and engaged in student teaching. Almost all the participants were seniors; a few were juniors or postbaccalaureate students. Therefore, what these prospective teachers knew, and the ways in which they thought, reflected the knowledge they had developed by the end of their formal mathematical study but before they had begun their teaching careers.

Instrumentation

Rationale

As noted earlier, this study had two phases. The first phase of the study aimed at getting a general picture of the prospective teachers' subject-matter knowledge and pedagogical content knowledge. The second phase aimed at clarifying this picture and adding more details. The instruments used were a questionnaire for the first phase and the same questionnaire and an interview for the second phase. The same questionnaire was used for both phases in order to enable us to generalize the findings from the second phase by comparing the answers to the questionnaires in both phases. Probing during the interview was designed to give a more accurate and detailed picture of the subjects' subject-matter content knowledge and pedagogical content knowledge. The probing focused on asking subjects to explain what they did on the questionnaire and why, asking for their reactions as teachers to students' misconceptions, and asking questions related to the questionnaire but requiring more general, longer, or more thoughtful responses. In this manner, the interview was used to clarify answers to the questionnaire and at the same time as an instrument to gather information on questions that were too difficult to answer via a written questionnaire. Both questionnaire and interview were specifically developed for this study.

Questionnaire

The questionnaire included two types of items: nine nonstandard problems addressing the different aspects of teachers' subject-matter knowledge about functions, and "students'" mistaken solutions or misunderstandings to be analyzed or responded to (6 items). Both types of items were based on students' limited conceptions and mistakes as described in the literature or known from the author's personal experience with students. The focus of this paper—the essential features of the concept of function: arbitrariness and univalence—was mainly addressed in the four items given in Figure 1 but was touched on, to some degree, by all the items in the questionnaire.

1. a) Give a definition of a function.

   b) A student says that he/she does not understand this definition. Give an alternate version that might help the student understand.

2. How are functions and equations related to each other ?

3. A student is asked to give an example of a graph of a function that passes through the points A and B (See Fig. 1).

   The student gives the following answer (See Fig. 2).

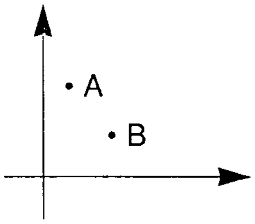


Figure 1

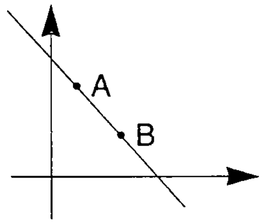


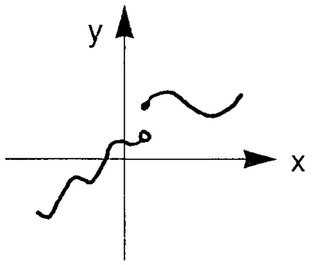
Figure 2

   When asked if there is another answer the student says: “No”.

   - If you think the student is right—explain why.
   - If you think the student is wrong—how many functions which satisfy the condition can you find? Explain.

4. A student marked all the following as non-functions. (R is the set of all the real numbers, N is the set of all the natural numbers).

   (i)



   (ii) f : R —> R
   f (x) = 4

   (iii) g : N —> R

   (iv) A correspondence that associates 1 with each positive number, −1 with each negative number, and 3 with zero.

   (v) $g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$

   (vi) {(1,4), (2,5), (3,9)}

   a) For each case decide whether the student was right or wrong. Give reasons for each one of your decisions.

(i) Right/wrong because	(ii) Right/wrong because
(iii) Right/wrong because	(iv) Right/wrong because
(v) Right/wrong because	(vi) Right/wrong because

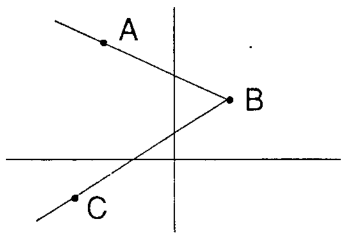
   b) In cases where you think the student was wrong, try to explain what the student was thinking that could cause the mistake.

Figure 1. Examples of questionnaire items.

Interview

The interview consisted of two parts. In the first part subjects were presented with items that required more general, longer, and more thoughtful responses than those of the questionnaire, as well as items that needed a flexible structure that only an interview can provide (see examples related to the focus of this paper in Figure 2). Also included were items about issues that emerged as important after analysis of the first-phase questionnaires. An issue was considered important if a relatively large number of subjects referred to it and it had a high degree of relevance to the learning and teaching of functions. For example, almost no one mentioned the vertical line test in response to the first part of Question 1 (Figure 1), whereas it was used frequently in the second part. Hence we included Question 2 in Figure 2 in the interview. This example is discussed in detail in the results section.

1. Give an example of a function.
2. Is it important to teach the vertical line test for graphs of functions to students?
   Why?
   (What is the line test? What would you teach to your students? How would you teach it? Can you give me an example?)
3. When asked to draw a graph of a function that passes through the points A, B and C, a student gave the following answer:
   What do you think about this answer?



   (Is it correct? Why?
   - If "yes"—Is there any other correct answer? Give me an example. Can you find another one? Why do you think the student gave this answer?
   - If "no"—What would you like to see as a correct answer? Give me an example. Another one. Why do you think the student gave this answer?)

Figure 2. Examples of interview items.

In the second part of the interview the interviewees were asked to reflect on their thinking and to explain and clarify their answers to the questionnaire. In addition to their free, voluntary explanations, the interviewees were also probed uniformly and nonuniformly. The uniform probes were presented to all subjects and were based on the analysis of the first-phase questionnaires. These probes represented themes that appeared in many of the written answers (such as arbitrary versus specific functions) and were meant to clarify ambiguities (such as the meaning of "many"—finite or infinite). The uniform probing also included presentations of ideas and misconceptions that were found in answers of the first-phase questionnaires, asking the subjects first to relate and evaluate them and then to explain what they think the "student" had in mind when he or she answered that way (see examples of probing in Figure 3). The nonuniform probing was based on the specific answers each subject gave to the questionnaire and was meant to clarify ambiguous answers and discover specific dimensions that seemed important.

(1)[a] - What did you assume the student didn't understand?
- Why is there, in the definition of functions, the requirement of having only one image for each element in the domain?

(2) - Are all functions equations? What do you mean?
- Are all equations functions? What do you mean?
- Can all functions be represented by an algebraic expression? a formula?

(3) - How many functions? (Can you give me some examples? More? Anything else?)
- How do you know that there is an infinite number of functions?

(4) - Do you think that there is a formula to describe the graph in (i)?
- Can you graph (v)?
- Many students said that (vi) is just a set of points and not a function; others said that it is not a function since 3 should go with 6. What do you think?

[a]Numbers in parentheses refer to the items given in Figure 1.

Figure 3. Examples of interview probing on the questionnaire.

Since the process of reflection that takes place during an interview causes learning (e.g., Confrey, 1987), questions that did not appear on the questionnaire were used before the discussion of the answered questionnaire. Nevertheless, the first item of the questionnaire that dealt with the definition of a function (Item 1, Figure 1), was discussed at the very end of the interview in order to eliminate its influence on the rest of the items.

Procedures

Data collection for this study was conducted from November 1987 to April 1988. The administration of the first-phase questionnaires took place in the regular mathematics methods class by the regular instructor of that class. Data collection for the second phase took place on two consecutive days. Each of the 10 subjects answered the questionnaire on one day and was interviewed the next day. This was done so that the subjects could easily recall their answers to the questionnaire and the reasoning behind them. The researcher administered the questionnaires and conducted the interviews in the second phase.

Subjects were not allowed to use any resources and had about an hour to complete the questionnaire. Interviews took about two hours each. Probing was an important component of the interview procedure. Some of the probes were standard—"Why?" "What do you mean by that?" and "Can you give me an example?" There were also times when the probes were specific to the given situation. The interview sessions were audiotaped and transcribed.

Data Analysis

Questionnaire

The analysis began by surveying about 40 questionnaires—a few from each of the seven first-phase universities—in order to gain an impression of the responses. This analysis led to the creation of preliminary categories of responses for each question and subquestion. An attempt was made to avoid imposing any preconceptions of what the answers might or should have been, in order to present as accurate a picture as possible of the different answers and to be open to unexpected directions.

These preliminary categories were then examined and modified after consulting experts in mathematics education. Similar categories were combined, and new categories that were considered important according to what was known about students' difficulties and limited conceptions of function were added. At the end of this process, each of the questions and subquestions of the questionnaire had a list of modified categories related both to the answers and the procedures used to obtain those answers and to themes that emerged from the different aspects of subject matter knowledge for teaching (Even, 1990).

After refinement, the final categories were used to analyze all the questionnaires. No attempt was made to reduce the number of categories at this stage, but rather category reduction was postponed until viewing the preliminary results. Experts in mathematics education were also consulted both at this stage and when faced with problematic cases. When there was disagreement about the categorization of the problematic cases, a discussion with several experts took place until agreement was achieved. In many cases an answer was categorized several times: by the final answer, by the method used to get that answer, by attention given to important issues (in relation to the different aspects of subject-matter knowledge), or by common issues (to many of the questionnaires).

Interviews

The interviews were transcribed in order to insure easy access to the data. The analysis began by listening to the taped interviews and editing the transcripts. Then, the interview transcripts were analyzed by person, by question, and by theme in the following manner. First, each person's answer to an interview question was summarized and significant comments were recorded. Then, a summary table for each interview question was created across subjects. Attention was given to several dimensions, such as the final answer, the method used to get that answer, mistakes made, ideas, preferences, ways of evaluating students' mistakes, and other significant comments. The above analyses were used for thematic analysis. The themes correspond to the subject-matter knowledge and pedagogical content knowledge aspects being studied.

RESULTS

The results reported here are results related to teacher subject-matter knowledge and pedagogical content knowledge with regard to arbitrariness and univalence. Arbitrariness is examined by referring to the arbitrary nature of the relationship between the two sets, which seems to be more crucial to teacher understanding of functions than arbitrariness of the sets. This section includes results of the analysis of data from both the first and the second phase of the study. The interview data are used to complete and clarify the questionnaire data as well as highlight new issues. For each feature (arbitrariness and univalence) the subjects' pedagogical content–specific choices are described and a comprehensive analysis of their related subject-matter knowledge is given. An illustrative case of the compartmentalization phenomenon is also included in the results section. Interpretations of the teachers' subject-matter knowledge and consequences to pedagogical content–specific choices are discussed in the discussion section.

Arbitrariness

The participants' definitions (or explanations) of a function (Item 1, Figure 1) were categorized as "modern" if there seemed to be some reference to the arbitrary nature of functions. (Correctness and accuracy of the definitions were not evaluated in the process of categorization.) For example: "A function is a set of ordered pairs (x, y) that have different x values but may or may not have the same y value." (Quotations from the questionnaires are reported in plain text; quotations from the interview are set off in dialog format.) Subjects' definitions were categorized as "old" if some regularity of the function behavior was included. For instance, "A function is a relationship between coordinates that meets certain requirements of smoothness, and...."

In Item 1 (Figure 1) the subjects were asked to give first a definition of a function and then an alternate version that might help a student with difficulties. Table 1 summarizes the function definitions that were given by the first-phase subjects, both with and without reference to a student. The relatively large number of 36 "others" in the "function definition for students" category is mostly due to subjects who did not give a definition or description of a function, but rather gave the student a rule to follow, such as the vertical line test. This issue is discussed in detail later in this article.

Table 1
Distribution of Function Definitions and Function Definitions for Students

	Modern	Old	Other	N/R	Total
Function definition without reference to a student	78	53	11	10	152
Function definition with reference to a student	27	67	36	22	152

Note. N/R = No response.

Pedagogical Content Knowledge: Defining a Function for Students

Table 1 shows that many prospective teachers who used modern terms when defining a function did not do so when approaching a student with difficulties. For example, Valerie defined a function as "a 1-1 mapping of a set of points X onto Y." But for the student, she did not use the term "mapping" and defined a function as an operation, which implies a nonarbitrariness characteristic: "You take a group of numbers. You perform some operation on the numbers (such as multiplying, etc.). This gives you a second group of numbers. The operation you did is called a function." In her interview, Valerie explained that she changed her definition for the student, since she assumed a student would not understand what a mapping is and would thus not be able to understand what she was talking about.

Subject Matter Knowledge: Conceptions of Arbitrariness

Three trends were detected in the subjects' responses: (1) functions are (or can always be represented as) equations or formulas, (2) graphs of functions should be "nice," and (3) functions are "known." These trends are described in the following.

Functions are equations and can always be represented by formulas. A large group of participants (33 of 53) whose definitions of a function (Item 1, Figure 1) were categorized into the old (specific) category defined a function as an equation, an algebraic expression, or a formula. For example, one subject wrote, "A function is an equation with a one-to-one correspondence between the variables." Another teacher candidate stated, "A function is a numerical expression that...."

All responses included in this category explicitly used one or another of the words "equation," "algebraic (numerical) expression," or "formula." However, even those who did not define a function as an equation may have been conceiving a function that way. This became apparent from the answers to Item 2 (Figure 1). When asked to describe the relationship between functions and equations, an additional 26 subjects (of the 33 who defined a function as an equation, formula, or expression in Item 1) said that functions are equations, or that rules for functions are equations (without any additional remarks that some functions may not be representable by equations). For example, one subject claimed, "They're the same thing." Another wrote, "A function is really an equation."

Although one may claim that describing functions as equations does not necessarily mean that formulas or expressions are involved, the interviews clarified that the subjects used the term "equations" to describe functions that are represented by an expression. Seven (of ten) second-phase subjects expressed the belief that *all* functions can be represented by using a formula. For example,

Bob: Yes, I think you could write all functions in terms of equations. It might be a trigonometric equation, like *sin x,* but in every term the y value is going to be equal to some operation with x value.

Researcher: So you can always describe a function using a formula or an equation?

Bob: I think so.

Altogether, about half the participants (who responded to both Items 1 and 2, Figure 1) showed signs of having equations as the dominant component of their concept image of functions. This tendency was also apparent from the examples of functions that the second-phase subjects gave during the interview (Item 1, Figure 2): eight out of ten started with algebraic expressions. They seemed to think of equations as the "real" functions.

Graphs of functions should be "nice." Some of the prospective teachers expected graphs of functions to be "nice" and smooth. Seven explicitly included such a requirement in the definition. For instance: "A function is a relationship between coordinates that meets certain requirements of smoothness, and...." Usually, however, the expectation that graphs of functions should be "nice" became apparent when the participant was faced with other situations. For example, when having to decide whether the following is a function (Item 4v, Figure 1),

$$g(x) = \{ \begin{array}{l} x, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number.} \end{array}$$

Brian justified his decision that it is a function by stating, "There is an assignment of a single value to each number." Later, when he sketched the graph during the interview, he got a few points on the x-axis for irrational numbers: π, $\frac{7}{4}$ [sic], $\sqrt{13}$; and a diagonal line $y = x$ with "holes" in it (see Figure 4).

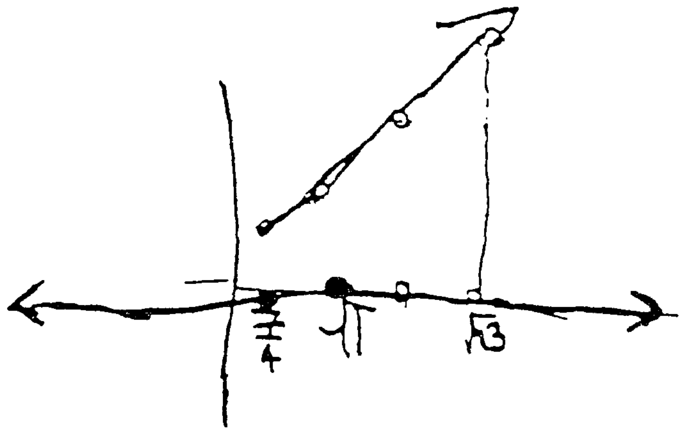


Figure 4. Brian's graph.

At this point he had a hard time deciding whether this graph is the graph of a function.

Brian: I don't know if it's a function. It fits the criteria of mapping, but it does not look pretty. It's not really graphable. It might just not be. But this is a discontinuous function. You're allowed to do discontinuous. There aren't sharp points. Oh, well...[shrugged his shoulders, could not make up his mind whether it was a function].

Another kind of "niceness" expectation that was confused with continuity in some peculiar way was not to have "sharp points" or "sharp corners" in a graph. Tracy illustrates this point in her response to Item 3 (Figure 2):

> But it's not continuous at the point B. So the function is not continuous, so it's not a function. Plus, it does not pass the vertical line test either. So, it's two points that, you know, make it fail being a function.[2]

It is clear from the data that at least 18 first-phase subjects expected graphs of functions to be continuous, as evidenced by their decision that the graph in Item 4i (Figure 1) is not a function because it is not continuous. It is not clear how many were willing to accept discontinuous graphs but still expected graphs not to be "too weird" or have "sharp corners," since only the interview data, which was limited to ten subjects, provided this information.

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[2]This confusion between continuous and differentiable functions is similar to the debate about functions and continuous functions during the 18th century (Kline, 1972). At that time, only differentiable functions were accepted as functions. Later, continuous (in the modern sense) but nondifferentiable functions were also accepted. Those functions were called discontinuous functions, referring not to discontinuity in the graph but rather to "discontinuity" in the expression that described the graph (several expressions instead of only one).

Functions are "known." When asked how many functions can pass through two given points (Item 3, Figure 1), most of the participants responded that there are infinitely many such functions. Only 12 subjects said that the number of functions is finite. However, many of those who said that the number of functions is infinite explained it by referring to specific examples of functions (which was sufficient to answer the question). They did not use the arbitrariness characteristic of functions. For example, one subject wrote "There are infinite parabolas that would satisfy the conditions."

Table 2 summarizes the answers of the first-phase subjects by the number of functions they thought could go through two points and by their explanation of their choice (i.e., whether they referred to arbitrary functions or used specific examples of functions only in their explanation).

Table 2
Distribution of Answers by the Number and Kinds of Functions That Can Pass Through Two Points

	Infinite	Finite	N/R	Total
Arbitrary	35	0	5	40
Specific	25	12	20	57
N/R	41	0	14	55
Total	101	12	39	152

Note. N/R = No response.

Giving specific examples (especially an infinite family, such as the parabolas above) without referring to arbitrary functions does not necessarily mean that the person is not aware of this aspect of functions. In order to investigate this further, the second-phase subjects were asked during the interview how many functions can pass through three noncollinear points and were probed more about both questions. The results revealed three different kinds of answers.

Some claimed that only a finite number of specific functions pass through two or three given points. Brian's answer illustrates this. He found two graphs that pass through the two given points, and on the basis of that, he said,

> I would guess just four. Just from the inference that I know two, there may be a few [more] out there.

Others said that there are infinitely many specific functions that pass through two or three points. This is illustrated by Bert's answer, in which he suggested different parabolas. Even after being pressed during the interview to find more functions, he did not come up with any. Some people found a broader class of functions and suggested, for example, an infinite number of polynomials. But again, when asked if there are other functions that may work, they said that they could not think of any. This kind of answer seems to indicate a belief that functions have specific names or shapes or are somehow "known" to be functions. By saying that they could not think of any other function, the participants seemed to say that there was not any other function *known* to them.

The third group of participants said that there are infinitely many arbitrary functions that pass through two or three points. Huey illustrates this by describing how, by free drawing, he can get infinitely many functions.

Huey: Really any function value, anything that is a function....Infinite, because I could just draw a million curves.

Researcher: Do you have names for all these curves? Are those specific curves?

Huey: No, there's no specific names. You couldn't even write equations for all of them that would satisfy...but certainly for some of them, like parabolas.

Most subjects answered the same way in both the two- and the three-point cases. But Brian, who started with a finite number of specific functions in the first case, changed his answer in the second case and said that the number of functions can be infinite: an infinity of parabolas and some other specific examples.

Univalence

Pedagogical Content Knowledge: The Vertical Line Test

While only 3 first-phase subjects (of the 142 who responded to Item 1, Figure 1) used the "vertical line test" in their definition of a function, 26 used it as an explanation for students, usually without explaining what a function is but only giving the student a test. For example: "By graphing the function and doing the vertical line test, a line never crosses the graph more than once."

A tendency to provide the student with the "vertical line test" as a rule that the student can follow and get the right answers (without needing to understand) was revealed during the interviews. The following excerpt illustrates this.

> If they're told to figure out whether it's a function or not, using the definition, they probably wouldn't be able to do it. If they know the vertical line test works, even if they don't know why it works, they can see right away why this is a function, because they can go through with a ruler or a straightedge and vertically go across the function, looking for places where there are two points.

The interviews also revealed that the "vertical line test" was chosen by the subjects because it was commonly used by their teachers when functions were taught. This is illustrated by the following excerpt:

> I'm sure everybody learns the same rule in high school. That you can draw a vertical line through the graph. And if it only intersects the graph at one point, then it's a function. If it intersects at more than one point, it is not a function.

Subject Matter Knowledge: Knowing How and Knowing Why

Familiarity with univalence. About half of the first-phase and almost all of the second-phase subjects included this requirement in their definition of a function. For example, "A function is a relation such that a number in the domain can only be matched to one number in the range." Six stated it backwards (i.e., for each element in the range there is only one element in the domain).

Between one-third and one-half of the participants used the univalence requirement as a criterion for checking whether given mathematical objects (Item 4, Figure 1) were functions. (The other methods used were based on a limited concept image of functions, such as discontinuity or familiarity with the specific function at hand; e.g., "This is a constant function.")

Why univalence? The interviews revealed that the univalence property was considered to be a very important characteristic of functions by the prospective teachers. But when asked during the interview to explain the importance of univalence, the subjects gave two kinds of immediate responses that showed that they did not really know. Eight subjects simply stated, "I don't know." Two subjects claimed that the role of univalence is to distinguish between relations and functions.

After being pressed to think of an explanation for this requirement (Why is it important to distinguish between functions and relations?), six subjects provided a greater variety of answers. Three tried to use everyday life, engineering, or science as the source of this requirement, making no connection to pure mathematics. They claimed that in those areas, in many cases, there was a need to have a single definite answer. Another three thought that the importance of the requirement was rooted in mathematics. Valerie was the only one who came close to the historical explanation of keeping things manageable.

> I suppose it makes it easier for all the other things you do, like for finding inverses, you'd have a more difficult time with it otherwise, probably.

Bert, who also thought that the univalence requirement came from mathematics, described the origin of the requirement as arbitrary.

> It seems like that would be, whoever decided to call that a function just made it one of the requirements. I would just think, that would be, whoever decided to call it a function just decided: if it looks like a graph, like this, and has only one, and I'm going to call that a function.

Brian's Case: Arbitrariness and Univalence

Brian's case illustrates the compartmentalization phenomenon. Brian's apparently conflicting schemes about arbitrariness and univalence resulted in inconsistent behavior as described in the following.

At first, it seemed that Brian understood the univalence requirement. He used it in his definition of a function (the emphases in the following quotations were added): "A thing which maps every element in a domain set onto another *unique* element in the range set." When asked to give an alternate version of this definition for a student who does not understand it, he also emphasized the univalence requirement: "For every number you put into a function you get *only one* number back out."

In addition to memorizing it, Brian used the univalence requirement correctly to decide whether objects were functions. For example, he decided that the following (Item 4v, Figure 1),

$$g(x) = \begin{cases} x, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number,} \end{cases}$$

is a function because "There is an assignment of a *single* value to each number." He also used the "vertical line test" to support his decision to accept a given graph (see Item 4i, Figure 1) as a function.

The first crack in this "ideal picture" occurred when Brian was asked during the interview to explain the "vertical line test," which he referred to as important to teach to students. He said,

> Like if I was going to have....Well. Uh, a circle is a function, but a circle doesn't pass the line test.

As the interview continued, he ignored the contradiction and continued to talk about the use of the "vertical line test" for deciding whether something is a function. Later, he was asked if he would accept the graph in Item 3 (Figure 2) as a function. The contradiction arose again:

> It doesn't pass the vertical line test, but a lot of functions don't do that.

Brian was asked to give an example of a function that passes through the three points A, B, and C. He sketched a graph of a function and said,

> It would pass the vertical line test.

Brian, as we saw earlier, could not decide whether a "weird looking" graph is a function, even though it satisfies the univalence requirement. At the same time, he accepted a circle as a function, although it does not satisfy univalence. In other words, for Brian the univalence property was neither a necessary nor a sufficient condition for a function.

Interestingly, Brian did not ignore the property as a few other subjects did. He used univalence to decide whether a given object is a function on several occasions, while at other times he accepted as functions relations that do not satisfy univalence. Brian's inconsistent behavior might be explained by conflicting schemes in his cognitive structure that were stimulated by different situations. At the end of the interview he admitted that he did not understand the use of univalence and claimed that it helps to understand what a function is when one first begins to study functions.

> It makes it a lot clearer up to a point. I've never been able to get it right in my own brain as to what it exactly is supposed to be. A function, well, early on it's a good idea to have one-to-one correspondence. Because really it's easy to see "that's a function," "that's not a function." Or to get the idea, it relates the idea of a function to an equation really well. But when you start moving out into concepts like cause-effect or things like that, it tends to break down. Like I have a problem trying to make an ellipse and call it a function based on my definition.
>
> It's great for the vertical line test. For linear functions it's perfect. For some others—quadratic—it's beautiful. But it just breaks down in a certain point. It works for linear functions. That's what it's there for. I think it's an over-generalized tool.

DISCUSSION

Arbitrariness

Many of the prospective teachers in this study did not hold a modern conception of functions. Rather, their conceptions were similar to students' conceptions as described in the literature (Dreyfus & Eisenberg, 1983, 1987; Lovell, 1971; Markovits et al., 1983, 1986; Marnyanskii, 1975; Thomas, 1975; Vinner, 1983; Vinner & Dreyfus, 1989). The subjects had specific expectations about functions and their behavior; for example, functions are (or can always be represented as) equations or formulas, graphs of functions should be "nice" ("weirdness" is subjective, of course), and functions are "known." In addition to the rejection of unfamiliar functions, there were also cases of using familiarity as a criterion of accepting nonfunctions as functions (this phenomenon is also described in Bakar & Tall, 1991). It seems that having a concept image of functions as equations with "nice" graphs made it "reasonable" to accept, for instance, the familiar circles and ellipses as functions.

An interesting finding of this study indicates that even though results of other studies (Markovits et al., 1983, 1986) might seem to imply that those who know that an infinite number of functions can pass through two or three points have better understanding about the arbitrary nature of functions than those who do not, there may not be such a big difference between the two. The ease with which a subject changed her or his answer from a finite number of specific functions in the two-point case to an infinite number of specific functions in the three-point case seems to indicate that the biggest difference lies between those who understand the arbitrariness characteristic of functions and those who do not, between those who expect functions to have specific names or shapes and those who do not.

Knowledge is derived from experience. It is not passively received but rather actively built up (von Glasersfeld, 1990). Therefore, students' concept image is determined by the functions they work with and not by the modern definition of a function that is presented to them, as several studies show (Bakar & Tall, 1991; Malik, 1980; Vinner, 1983; Vinner & Dreyfus, 1989). It seems that this was also the case with the subjects in this study. Most of the functions that prospective secondary mathematics teachers encounter during their mathematics courses are the kind that have a "nice" graph and can be described by an expression. The need for arbitrariness arises in only one mathematics course in preservice education—advanced calculus. But as the results of this study show, such brief experience is not enough—even subjects who have taken this course did not have a modern concept of a function. Therefore, it should not surprise us that when describing what a function is for students, many subjects chose to use their concept image and tended not to employ modern terms. So even if the decision not to use a modern definition for students might seem a good pedagogical decision, it should be attributed to the incomplete concept image of a function that many of the prospective teachers hold and not necessarily to mature pedagogical reasoning.

Univalence

Univalence is usually presented to students as one of the most important characteristics of functions, and this is what many of the subjects thought. They knew that it distinguishes between relations that are not functions and those that are. But without prompting, none of the subjects could come up with a reasonable explanation for the need of univalence. Even after prompting, most did not know why it is important to distinguish between functions and nonfunctions. These results should not surprise us because most textbooks (e.g., Coxford & Payne, 1987; Dolciani, Sorgenfrey, Brown, & Kane, 1986; Keedy, Bittinger, Smith, & Orfan, 1986; Nichols et al., 1986), and therefore, we may assume, many teachers, do not provide students with examples or activities that would help them understand what one can do with functions that one cannot do with relations that are not functions. Also, it is possible, perhaps likely, that many teachers, as well as teachers of teachers, do not explicitly know (or never thought about) the need for this requirement.

Not knowing why univalence is needed may influence pedagogical content-specific choices, by making it "reasonable" to present students with easy procedures that overemphasize procedural knowledge without concern for meaning. This is exactly what many of the prospective teachers in this study did when choosing to provide students with the "vertical line test" as a rule to follow and get the right answers without concern for understanding.

CONCLUSION

A situation in which secondary teachers at the end of the 20th century have a limited concept image of function similar to the one from the 18th century is problematic. Based on a constructivist approach to learning, the teacher's main function, as Noddings (1990) and Confrey (1990) state it, is to establish mathematical environments that encourage exploration and strong acts of construction, environments in which students are encouraged to explore and raise questions (as is advocated also by the National Council of Teachers of Mathematics, 1991). When doing so, teachers may be put in situations where they have to deal with unfamiliar instances. Their pedagogical decisions—questions they ask, activities they design, students' suggestions they follow—are based, in part, on their subject-matter knowledge. Therefore, it is important that teachers develop a modern concept image of function.

In addition to the pedagogical arguments described above, there are also cultural arguments. The concept of function has changed over the years, not because someone arbitrarily decided to make changes, but rather because new knowledge in mathematics created the need for those changes. New developments created new branches of mathematics that also led to changes in the definitions. Mathematics teachers who are constrained by their limited and underdeveloped concept image may also be prevented from understanding current mathematics that is based on a more modern conception of functions. This situation may contribute to the cycle of discrepancies between concept definition and concept image of functions in students, keeping the students' concept image of a function similar to the one in the 18th century.

Still, having a modern concept image of functions is not sufficient. Teachers need to know, for example, *why* there is a need for univalence in the definition. Being familiar with the historical development of functions may help because it explains why functions came to be defined the way they are today and thus gives meaning to the definition. Having a meaningful understanding of the mathematical need of univalence can help teachers make knowledgeable decisions about the place of the concept of function in the curriculum and the emphasis they should put on the univalence requirement.

One immediate conclusion of this study is that an important step in improving teaching should be better subject-matter preparation for teachers. However, better subject-matter preparation does not mean changing the number of courses prospective teachers must take. Instead, mathematics courses should be constructed differently, in line with the constructivist views on teaching and learning (e.g., Confrey, 1990), so that better, more comprehensive and articulated understanding and knowledge of functions (and mathematics) is developed. Teachers need to have learning environments that foster powerful constructions of mathematical concepts. Unfortunately, the present mathematics courses teachers typically experience do not provide such an environment.

Good subject-matter preparation for teachers is necessary but not sufficient. Teachers tend to follow their own teachers' footsteps (Ball & McDiarmid, 1990) unless they have developed a different repertoire of teaching skills. This was illustrated in this study by the subjects' choice of the "vertical line test" as a representation of a function they would use as teachers. Developing a powerful teaching repertoire is part of pedagogical reasoning—the process of transforming subject-matter knowledge into forms that are pedagogically powerful (Shulman, 1987)—and depends, in addition to strengthening subject-matter knowledge, on the integration of different domains of knowledge (Ball, 1988). Therefore, a good content-specific pedagogical preparation is also needed.

A powerful content-specific pedagogical preparation based on meaningful and comprehensive subject-matter knowledge would enable teachers to teach in the spirit envisioned in the *Professional Standards for Teaching Mathematics* (NCTM, 1991). Teachers with this sort of preparation would be able to create learning environments for their students that foster the development of students' mathematical power. They would also be prepared to pose tasks for their students that are based on sound and significant mathematics and to help students develop a coherent framework for their mathematical ideas.

REFERENCES

Bakar, M., & Tall, D. (1991). Students' mental prototypes for functions and graphs. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the PME* (Vol. 1, pp. 104–111). Assisi, Italy.

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring with them to teacher education.* Unpublished doctoral dissertation. Michigan State University, East Lansing, MI.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching, Vol. 2* (pp. 1–48). Greenwich, CT: JAI Press.

Ball, D. L., & McDiarmid, G.W. (1990). The subject matter preparation of teachers. In W.R. Houston (Ed.), *Handbook of research on teacher education* (pp. 437–449). New York: Macmillan.

Bennett, A. A. (1956). Concerning the function concept. *Mathematics Teacher, 49,* 368–371.

Bottazzini, U. (1986). *The higher calculus: A history of real and complex analysis from Euler to Weierstrass* (Warren Van Egmond, Trans.). New York: Springer-Verlag.

Buchmann, M. (1984). The priority of knowledge and understanding in teaching. In J. Raths & L. Katz (Eds.), *Advances in teacher education* (Vol. 1, pp. 29–48). Norwood, NJ: Ablex.

Confrey, J. (1987). The constructivist. In J.C. Bergeron, N. Herscovics, & C. Kieran (Eds.). *Proceedings of the 11th International Conference for the PME* (Vol. III, pp. 307–317). Montreal, Canada.

Confrey, J. (1990). What constructivism implies for teaching. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (Journal for Research in Mathematics Education: Monograph Number 4, pp. 107–122). Reston, VA: National Council of Teachers of Mathematics.

Coxford, A. F., & Payne, J. N. (1987). *Algebra 1 revised edition.* Orlando, FL: Harcourt Brace Jovanovich.

Dolciani, M. P., Sorgenfrey, R. H., Brown, R. G., & Kane, R. B. (1986). *Algebra and trigonometry, structure and method book 2.* Boston, MA: Houghton Mifflin.

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness & periodicity. *Focus on Learning Problems in Mathematics, 5*(3 & 4), 119–132.

Dreyfus, T., & Eisenberg, T. (1987). On the deep structure of functions. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.). *Proceedings of the 11th International Conference for the PME* (Vol. I, pp. 190–196). Montreal, Canada.

Even, R. (1989). Prospective secondary teachers' knowledge and understanding about mathematical functions (Doctoral dissertation, Michigan State University, 1989). *Dissertation Abstracts International, 50,* 642A.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21,* 521–544.

Even, R., & Markovits, Z. (in press). Teachers' pedagogical content knowledge of functions: characterization and applications. *Structural Learning.*

Eves, H. (1983). *Great moments in mathematics (After 1650)* (pp. 153–155). Washington, DC: Mathematical Association of America.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: D. Reidel.

Hamley, H. R. (1934). *Relational and functional thinking in mathematics: The 9th Yearbook of NCTM.* New York: Bureau of Publications, Teachers College, Columbia University.

Hight, D. W. (1968). Functions: Dependent variables to fickle pickers. *Mathematics Teacher, 61*(6), 575–579.

Keedy, M. L., Bittinger, M. L., Smith, S. A., & Orfan, L. J. (1986). *Algebra.* Menlo Park, CA: Addison-Wesley.

Kennedy, M. M. (1990). *A survey of recent literature on teachers' subject matter knowledge* (Issue Series 90-3). East Lansing: Michigan State University, National Center for Research on Teacher Education.

Kieran, C. (in press). Functions, graphing, and technology: Integrating research on learning and instruction. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of function.* Hillsdale, NJ: Lawrence Erlbaum.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *College Mathematics Journal, 20*(4), 282–300.

Kline, M. (1972). *Mathematical thought from ancient to modern times.* New York: Oxford University Press.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction, 3,* 305–342.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77,* 247–271.

Lovell, K. (1971). Some aspects of the growth of the concept of a function. In M. F. Rosskopf, L. P. Steffe, & S. Taback (Eds.). *Piagetian cognitive development research and mathematical education* (pp. 12–33). Reston, VA: National Council of Teachers of Mathematics.

Malik, M. A. (1980). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematics Education in Science and Technology, 11,* 489–492.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1983). Functions: linearity unconstrained. In R. Hershkowitz (Ed.), *Proceedings of the seventh international conference of PME* (pp. 271–277). Israel: Weizmann Institute of Science.

Markovits, Z., Eylon, B., & Bruckheimer, M. (1986). Functions today and yesterday. *For the Learning of Mathematics, 6*(2), 18–24, 28.

Marnyanskii, I. A. (1975). Psychological characteristics of pupils' assimilation of the concept of function. In J. Kilpatrick, I. Wirszup, E. Begle, & J. Wilson (Eds.), *Soviet Studies in the Psychology of Learning and Teaching Mathematics XIII* (pp. 163–172). Chicago, IL: SMSG, University of Chicago Press. (Original work published 1965.)

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics.* Reston, VA: Author.

Nichols, E. D., Edwards, M. L., Garland, E. H., Hoffman, S. A., Mamary, A., & Palmer, W. F. (1986). *Algebra 2 with trigonometry.* New York: Holt, Rinehart & Winston.

Noddings, N. (1990). Constructivism in mathematics education. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (Journal for Research in Mathematics Education: Monograph Number 4, pp. 7–18). Reston, VA: National Council of Teachers of Mathematics.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher. 15*(2), 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1–22.

Tamir, P. (1987). *Subject matter and related pedagogical knowledge in teacher education.* Paper presented at the annual meeting of the American Association for Educational Research, Washington, DC.

Thomas, H. L. (1975). The concept of function. In M. E. Rosskopf (Ed.), *Children's mathematical concepts: Six Piagetian studies in mathematics education* (pp. 145–172). New York: Teachers College Press.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematics Education in Science and Technology, 14,* 293–305.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20,* 356–366.

Vinner, S., Hershkowitz, R., & Bruckheimer, M. (1981). Some cognitive factors as causes of mistakes in the addition of fractions. (1981). *Journal for Research in Mathematics Education, 12,* 70–76.

von Glasersfeld, E., (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (Journal for Research in Mathematics Education: Monograph Number 4, pp. 19–30). Reston, VA: National Council of Teachers of Mathematics.

Wilson, S. M., Shulman, L. S., & Richert, A. (1987). "150 ways of knowing": Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teacher thinking* (pp. 104–124). Sussex: Holt, Rinehart, & Winston.

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