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Historical and pedagogical aspects of the definition of function†

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Since the beginning of the nineteenth century, mathematics has been undergoing a revolution. In this paper the author discusses the definition of the function in the light of this revolution.

"The greatest threat to the life of mathematics is posed by the mathematicians and their most potent weapon is their pedagogy."—Morris Kline.

Since the beginning of the nineteenth century, mathematics has been undergoing a revolution. Over this period mathematics has acquired a new spirit and axiom, rigour and abstract concepts are now the main components of mathematics. Prior to this change, it had been growing in conformity to its surroundings and environment, but now it grows in response to a vigorous inner drive. Mathematics no longer has the status of the queen of the sciences; it now has its own kingdom. These new developments, originally confined to the circle of research mathematicians, started making their way in the university teaching by the middle of this century. The university mathematics curricula were gradually modified. The corresponding change in school mathematics came a little later, but when it came, it came suddenly. A part of the traditional mathematics was considered outmoded and removed while several new topics, advertised as being more useful and more meaningful, representing the spirit of new math, have been added. Such a move was inevitable. But unfortunately the resulting vacuum was hastily filled by intuitive geometry, set theory and algebraic operations. Even the teaching of mathematics took on the mood of algebraic treatment. In a strange mixture of rigorous and sloppy definition, completeness was listed in the capsule of axioms in defining real numbers. And function as an ordered pair became an obligatory part of school mathematics.

When mathematics had been in a process of change, the pre-revolutionary mathematics gained an importance in the social milieu. Society realized that calculus and its off-shoots, such as statistics, were useful for other areas of knowledge. The scholars of other disciplines took to advising their students to learn calculus. In response to the advice, students of the biological and social sciences, of commerce as well as of humanities took mathematics in their programme. This has resulted in a tremendous increase in the enrolment in the calculus course over the last two decades.

The increasing demand for calculus courses gave a relative importance to the courses which serve as a prerequisite to calculus. In practice the function course became the most important and fundamental course in the mathematics programme.

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Intoxicated by their success, mathematicians paid little attention to the needs or future needs of students and went on designing and teaching the function course laced with set-theoretic notations and other abstract concepts so as to enlighten the students with the new spirit mathematics has achieved. Whereas the students need to learn function in order to understand the techniques of calculus, the scholars from non-mathematical areas also expect their students to know pre-revolutionary mathematics rather than acquiring vague ideas about abstract concepts clouded with notations and axioms. Of course there is no harm in learning a little more than you need, but because of conflicting goals both teachers and students confront insurmountable difficulties. For a student, mathematics has now become harder to learn than it was before. Calculus is not an easy subject to learn or to teach and when concepts not directly related to its central theme are added to its prerequisite, the subject becomes out of reach for average students. The prerequisite function course has its own pedagogical problem. The teachers engaged in teaching the function course face enormous difficulties in communicating this abstract concept in the classroom and the attempts of mathematics educators in designing and redesigning this course have not yet led us to any consensus. Moreover, the necessity of teaching the modern definition of function at school level is not at all obvious and most of the instructors feel that pedagogical considerations were ignored while designing the course content and the mode of presentation.

A revision of the function course seems imperative. Why the students learn this course and what we want them to achieve out of this course should be re-evaluated. In this discussion, the history of function should also have an input. It tells us when, why and in what form the concept of function entered mathematics and how we arrived at this abstract concept. In fact, the modern definition of function appeared on the horizon of mathematics late in the history of the discipline in order to study more advanced mathematics, rather than provide a starting point for a pre-calculus course.

The concept of function originated when Galileo (1564–1642) proposed a programme for the study of motion [1]. A mathematical study of motion was not in the frame of reference of the Arabs, while the Greek mathematicians never entertained the concept of speed. The investigation of a relation between two varying quantities had been fundamental in arriving at the concept of function. With the analytic geometry of Descartes (1637), curves described by motion or formula referring to motion rather than by construction, were included in investigations and a relation representable in expression and its graph were now accepted as mathematical objects. The invention of calculus reinforced this trend of thinking. Over the next two centuries, Euler, the Bernoullis and other mathematicians developed calculus to deal with physical problems. For these mathematicians, a function was an analytic expression representing the relation between two variables with its graph having no corners; this is usually referred to as Euler's definition of function. Any attempt to widen the class of functions was not readily approved mainly because calculus did not require any more sophistication than the classical definition. A survey of problems and a pedagogically acceptable theory for a first course in calculus shows that Euler's definition covers all the functions used or required in the course. In other words, one can successfully design a calculus course based on Euler's function as its foundation. In fact, throughout the calculus course, one never confronts a situation where one has to use the modern definition of function. Only its particular form is used. A student retains a concept only if it is used in the course; if only its particular form is used, the student unconsciously accepts the particular form as the definition. That is why, at the termination of a calculus course, the student understands function as a smooth relation between two varying quantities.

The modern concept of function was introduced by Dirichlet to study not calculus but more advanced concepts. In the early 19th century, inspired by the problem of heat conduction in a rod, Fourier proposed to study the convergence of 'Fourier series'. This interesting problem soon attracted the attention of mathematicians and in 1829, Dirichlet made a breakthrough when he discovered the conditions on the function f so that its Fourier series converges to f. He also presented an example of a function that does not satisfy the conditions, i.e. $f(x)=0$ when x is rational and $f(x)=1$ when x is irrational [2]. In his studies, Dirichlet realized that, in fact, what we are interested in is that for each x the series converge to a real number which is the value of f at x. This problem would be difficult to study if we were to stick to the classical definition of function because then one must consider the convergence of the series as well as the variation of x simultaneously. Thus Dirichlet re-defined function: y is a function of x if for any value of x there is a rule which gives a unique value of y corresponding to x.

This new concept at first met with criticism. It was too general, Chebychev argued, to be useful in Analysis; such a function is pathological. For almost a century, mathematicians including Baire, Borel and Levesque debated the question of how a function should be defined [5]. There was a search under way to arrive at a definition not as restricted as of Euler's but useful in Analysis. However, whatever departure from the classical definition was entertained, the Fourier coefficients could no longer represent an area under a curve. This trend of thought led to new research in the trigonometric series and the theory of integration. A new spirit and a new dimension in mathematical thinking created a subject—Analysis—derived from Calculus but different from Calculus. In return, Dirichlet's definition of function received an important reinforcement. With the introduction of the concepts of metric space and topology, it was realized that the properties of a function depend on the structure of the sets on which it is defined and where it takes its values. This led to the concepts of domain and range; and the function further escalated to a higher level of an abstract concept. In 1917, Caratherdory defined a function as a rule of correspondence from a set A to real numbers and in 1939 Bourbaki defined function as a rule of correspondence between two sets and in later chapters observed that it is a subset of the cartesian product of sets [3], [4]. By the end of the first half of this century, the Dirichlet–Bourbaki definition of function had become established as textbook terminology.

One may remark that the problem of a vibrating string should also receive credit for the new definition of function. But such an assertion does not seem to be entirely convincing. Much earlier than Dirichlet, Bernoulli did say that the function f in the problem of a vibrating string must be expressible in the form of a series with terms such as $\sin n\pi x$; but his contention was flatly rejected by most of the contemporary mathematicians including Euler, D'Alembert and Lagrange [5]. Had mathematicians accepted the Bernoulli's contention, the kingdom of mathematics might have widened a century earlier. This rejection, however, was not due to any innate conservatism or lack of vision. There was no pressing need that $f(x)$ represent anything else than the shape of a string. One can further suppose that the physical constraints in these two problems are of a different nature. The shape of a string fixed at both ends is visible whereas the temperature distribution in a rod can be imagined (or experimentally obtained) but cannot be seen. In the heat problem, the thinking is free from geometrical perception and this gives a freedom not conceivable when dealing with the string problem. The studies related to the heat conduction provided reasons for abandoning the classical frame of reference and resulted in the extension of the class of functions due to Dirichlet. Incidentally, this is evidence for the conclusion that in mathematics research a new concept receives recognition only when its relevance to the current phase of research is established. Then why should relevance not be taken into account when teaching a mathematics course?

There is still another consideration. The modern definition is algebraic in its spirit. It appeals to the discrete faculty of thinking and lacks a feel for the variable. Whereas, for calculus and other practical sciences the requisite training should enable the student to develop a feel for smooth change of the variables in phenomena. These are two different frames of thought and it is not obvious how one helps in understanding the other, particularly at the elementary level.

In conclusion, we note that the definition of function as an expression or formula representing a relation between variables is for calculus or a pre-calculus course; is a rule of correspondence between reals for analysis; and a set theoretic definition with domain and range is required in the study of topology. Since only a small percentage of school students eventually study analysis and topology, the set theoretic definition could be postponed to the beginning of these courses and a simple and easily understandable definition should be taught at the elementary level. One should also note that there is not yet any convincing evidence that a student introduced to an idea with a level of rigour and generality will develop a stronger liking for the subject or will be better trained for assimilating the techniques and concepts where only a particular form of ideas is in use.

It is indeed time for school teachers to participate in designing a function course so the entire responsibility is not left to mathematics educators and university research scholars.

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