IAN 2019

1) $\gamma' = (1-\sqrt{3}\cos t,\ -2\sin t,\ \sqrt{3}+\cos t)$

$\gamma'' = (\sqrt{3}\sin t,\ -2\cos t,\ -\sin t)$

$\gamma''' = (\sqrt{3}\cos t,\ 2\sin t,\ -\cos t)$

$$\gamma' \times \gamma'' = \begin{vmatrix} i & j & k \\ 1-\sqrt{3}\cos t & -2\sin t & \sqrt{3}+\cos t \\ \sqrt{3}\sin t & -2\cos t & -\sin t \end{vmatrix}.$$

$$= (2\sin^2 t + 2\sqrt{3}\cos t + 2\cos^2 t)\, i - (-\sin t + \cancel{\sqrt{3}\sin t\cos t} - 3\sin t - \cancel{\sqrt{3}\sin t\cos t})\, j$$

$$+ (-2\cos t + 2\sqrt{3}\cos^2 t + 2\sqrt{3}\sin^2 t)\, k$$

$$= (2+2\sqrt{3}\cos t)\, i + 4\sin t\, j + (2\sqrt{3} - 2\cos t)\, k$$

(4)

$$|\gamma' \times \gamma''| = 4 + 4\cdot 3\cos^2 t + 4\cdot \cancel{2\sqrt{3}\cos t} + 16\sin^2 t + 4\cdot 3 + 4\cos^2 t - \cancel{8\sqrt{3}\cos t}$$

$$= 4 + 16 + 12 = \sqrt{32} = \sqrt{4^2 \cdot 2} = 4\sqrt{2}.$$

$|\gamma'| = \sqrt{8} = 2\sqrt{2}$

$$\boxed{K = \frac{4\sqrt{2}}{(2\sqrt{2})^3} = \frac{1}{4}}.$$

(4)

$$\begin{vmatrix} 1-\sqrt{3}\cos t & -2\sin t & \sqrt{3}+\cos t \\ \sqrt{3}\sin t & -2\cos t & -\sin t \\ \sqrt{3}\cos t & 2\sin t & -\cos t \end{vmatrix} = \dots = 8.$$

$$\boxed{\tau = \frac{8}{(4\sqrt{2})^2} = \frac{1}{4}}.$$

(4)

2) Από υπόθεση $\langle \gamma(s), \gamma(s)\rangle = R^2 \Rightarrow$

$$\Rightarrow 2\langle \gamma', \gamma\rangle = 0 \Rightarrow \langle \gamma'', \gamma\rangle + \underbrace{\langle \gamma', \gamma'\rangle}_{1} = 0$$

$$\Rightarrow \langle \gamma'', \gamma\rangle = -1$$

Άρα $1 = |-1| = |\langle \gamma'', \gamma\rangle| \le \|\gamma''\|\cdot\|\gamma\| = k(s)\cdot R$

$$\Rightarrow \frac{1}{R} \le k(s)$$

(20)

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3) Έστω $f(x,y,z) = x^3 - y^3$ $\quad M_c = f^{-1}(c)$

$\nabla f = (3x^2, -3y^2, 0) = 0 \iff \boxed{x = y = 0,\ z \in \mathbb{R}.}$

• Για $c \neq 0$ τα το $(0,0,z)$ δεν ικανοποιεί την $f = c$.
Άρα $g^{-1}(c)$ είναι κανον. επιφ.

• Αν $c = 0$, τότε $x^3 - y^3 = 0 \Rightarrow x = y$, επίπεδο του $\mathbb{R}^3$, άρα είναι κανον. επιφ. (10)

Άρα $\forall c \in \mathbb{R}$

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4) ~~(α) 1ος τρόπος $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{9} - 1$, $M = g^{-1}(1)$ Άρα επιφάνεια~~
~~$\nabla f = 0 \iff (x,y,z) = 0$, $(0,0,0) \notin g^{-1}(1)$~~

Λάθος

~~2ος τρόπος~~ • X 1-1 (5)

• X κανονική $X_u \times X_v = \left(\frac{6u}{\sqrt{1-u^2-v^2}}, \frac{6v}{\sqrt{1-u^2-v^2}}, 4\right) \neq (0,0,0)$

$X_u = \left(2, 0, \frac{-3u}{\sqrt{1-u^2-v^2}}\right)$, $X_v = \left(0, 2, \frac{-3v}{\sqrt{\cdots}}\right)$

β) i) $X(0,0) = (0,0,3) = P$,

Άρα μια βάση του T_PM είναι η $\{X_u(0,0), X_v(0,0)\}$
$= \{(2,0,0), (0,2,0)\}$ (10)

$(2,-1,0)_P = 1\cdot X_u + \frac{1}{2}X_v \quad \therefore \quad Z \in T_PM$. (ή $Z \perp X_u(0,0)\times X_v(0,0)$) (10)

ii) Έστω $\bar{\gamma}(t) = (0,0) + t(1, -1/2) \in D$.

Θέτω $\boxed{\gamma(t)} = X \circ \bar{\gamma}(t) = \boxed{\left(2t, -t, 3\sqrt{1 - t^2 - \frac{t^2}{4}}\right)}$

Τότε $\gamma(0) = (0,0,3) = P$

$$\gamma'(0) = \left(2, -1, \frac{-15t/4}{\sqrt{1 - 5t^2/4}}\right)\Bigg|_{t=0} = (2,-1,0) = Z.$$

(10)

γ) $X_u = \left(2, 0, \dfrac{-3u}{\sqrt{1-u^2-v^2}}\right)$ $\quad X_u(0,0) = (2,0,0)$

$X_v = \left(0, 2, \dfrac{-3v}{\sqrt{1-u^2-v^2}}\right)$, $\quad X_v(0,0) = (0,2,0)$

$\therefore$ $\boxed{E = 4}$, $\boxed{F = 0}$, $\boxed{G = 4.}$

$N(0,0) = (0,0,1)$

$X_{uu} = \left(0, 0, \dfrac{-3\sqrt{1-u^2-v^2} + 3u\dfrac{(-2u)}{\sqrt{1-u^2-v^2}}}{1-u^2-v^2}\right)$, $X_{uu}(0,0) = (0,0,-3)$

$X_{uv} = (0,0,0)$ $\quad$, $X_{vv} = \left(0, 0, \dfrac{--}{--}\right)$, $\quad X_{vv}(0,0) = (0,0,3)$

Αρα $\quad e = \langle X_{uu}, N \rangle = -3$, $\quad f = 0$, $\quad g = -3$ (15)

Οποτε $\quad S_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -3/4 & 0 \\ 0 & -3/4 \end{pmatrix}}$

δ) Η μοναδική ιδιοτιμή του S_p ειναι $-3/4$.

Αρα $\boxed{k_1 = k_2 = -3/4.}$ (2) $\quad$ όλες οι διευθυνσεις ειναι (4) κύριες διευθυνσεις.

$K = k_1 k_2 = 9/16$, (2) $\quad H = \dfrac{-3/4 - 3/4}{2} = -3/4.$ (2) $\quad \boxed{10}$

5) Αν υπηρχε τέτοιο σημειο p, τοτε

$K(p) = 20 = k_1(p) k_2(p)$ $\quad H(p) = 1 = \dfrac{k_1(p) + k_2(p)}{2}$

Τοτε $(k_1 + k_2)^2 = 4 = k_1^2 + k_2^2 + 2k_1k_2 = k_1^2 + k_2^2 + 40$

Συνεπως $\quad k_1^2(p) + k_2^2(p) = -36 < 0$ $\quad \boxed{8}$

άτοπο