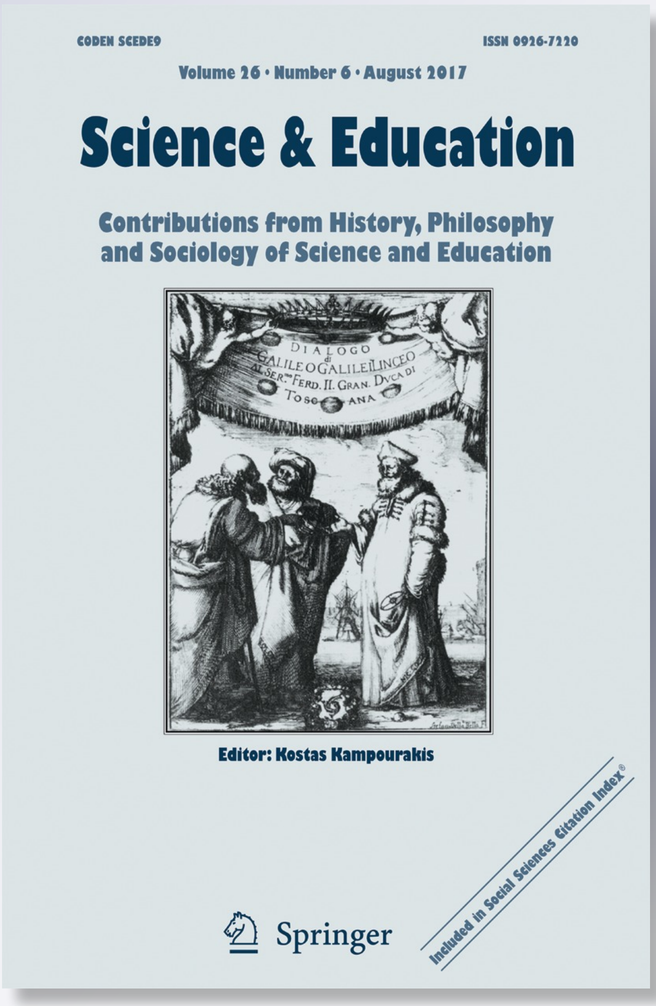
“There is One Geometry and in Each Case There is a Different Formula”

Ioannis Rizos, Anastasios (Tasos) Patronis & Dionyssios Lappas

Science & Education
Contributions from History, Philosophy and Sociology of Science and Mathematics

ISSN 0926-7220
Volume 26
Number 6

Sci & Educ (2017) 26:691-710
DOI 10.1007/s11191-017-9915-1



Springer

Sci & Educ (2017) 26:691–710
DOI 10.1007/s11191-017-9915-1

ARTICLE

"There is One Geometry and in Each Case There is a Different Formula"

Students' Conceptions and Strategies in an Attempt of Producing a Minkowskian Metric on Space-Time

Ioannis Rizos[1] · **Anastasios (Tasos) Patronis**[1] · **Dionyssios Lappas**[2]

Published online: 7 September 2017

Abstract In this paper, we analyze two episodes from an inquiry-based didactical research; the complete analysis of our research data is still ongoing. By taking into consideration various developments from the history of the geometry of space-time, our general aim is to explore high school students' conceptions about measurement of length and time in relatively moving systems, and lead the students to reconsider these conceptions in an attempt of constructing a new metric for space-time. The episodes are extracted from long (focused) interviews with two couples of students, based on a carefully designed fictional scenario. Two main strategies have been identified and are analyzed in the paper: one of them relies on imagination and intuition; the other one makes use of preexisting school mathematical knowledge, in arriving to a simplified formula of a Minkowskian metric.

1 Introduction: Epistemological Background and Aim of the Research

1.1 Children's and Young Students' Ideas About Motion in Space

Some researchers have noted parallels between young students' way of thinking and the development of science, in the sense that many of children's conceptions appeared in the past in natural scientists' thinking (Eckstein and Kozhevnikov 1997). For example, there is a similarity between the children's way of thinking and the Aristotelian theory of motion (Driver et al. 1985).

✉ Ioannis Rizos
rizos@master.math.upatras.gr

1 Department of Mathematics, University of Patras, 26110 Patras, Greece

2 Department of Mathematics, National and Kapodistrian University of Athens, 15784 Athens, Greece

However, as far as *motion in space* is concerned, some students' thinking is probably closer to Aristotle rather than Newton (Whitaker 1983), while some other students are reported to have formed strong beliefs in accordance with the Newtonian and Galilean mechanics (Eckstein and Shemesh 1993). Students' empirical–sensational reasoning may agree with common sense, but usually it is difficult to embrace modern scientific conceptions, as well as rigorous mathematical proofs. Therefore, for young students, *space* is experiential, so *homogeneous* and *isotropic*. However, as it will follow from our analysis, students' conceptions of space and motion may be intimately related to a context of *medium scale distances* and *low velocities*.

Students' ideas and interpretations are often formed not only as a result of everyday experience but also under the influence of the popular culture and of the attempts of popularizing knowledge in school textbooks. For example, people believe that Newton conceived the idea of gravity after the falling of an apple on his head, and that Galileo was inspired by simply observing natural phenomena. As Matthews wrote, in the case study of the pendulum motion (Matthews 1994, p. 112), "contrary to the textbooks, it was not observation but mathematics, and experiment guided by mathematics, that played the major role in Galileo's discovery and proof …."

1.2 Euclidean Metric Geometry and "Timeless" Geometrical Conceptions

From its early creation as a discipline, geometry has been founded upon axioms which refer to "points," "straight lines," and "angles." The emerging question, here, is about the relation of these entities to empirical knowledge resulting from observations. According to a well-known statement of Henri Poincaré (1905/1952, p. 49) "we do not make experiments on ideal lines and circles; we can only make them on material objects." This shows a tension between an axiomatically founded geometry, with axioms given a priori, and an experiment-oriented development which imposes restrictions and allows reforming examinations of the *Hypotheses which lie at the foundations of geometry*, as suggested by the title of the famous Riemann's 1854 habilitation.

The origins of Metric Geometry are traced back into the roots of cultural evolution. The mental consideration of an orthogonal triangle incorporates the embodied notion of *perpendicularity*, arising from the human perception of the vertical-horizontal spatial relations. Since then, the quadratic character of the metric has remained unchanged, until now, as an archetypal Pythagorean form (Lappas and Spyrou 2006). Thus, in antiquity, deep mathematical questions were already posed with respect to the nature of *magnitudes* and their measurement, such as incommensurability and the exact formulation of physical laws. However, as Felix Klein had observed, the concept of *motion*, was "consistently avoided" in ancient times, "as it frequently still is" (in modern times), and this fact "was due, in part, to philosophical considerations (…) It was feared that motion would bring into geometry an element foreign to it, namely, the notion of time" (Klein 1908/2004, p. 174).

In addition to epistemological difficulties due to the historical development of geometry, the teaching of geometry does not often lead the students into interrogation and inquiry. We refer here especially to the influence that school geometry has on students' conceptions of space, as well as on students' understanding of axioms in geometry (Patronis 1994; Kaisari and Patronis 2010). Greek students come across the Pythagorean theorem at an early stage of their studies, but in various degrees of abstraction and rigorousness. Apart from a standard formulation, the theorem's exposition (or proof) depends upon the age level of the mathematics class. In some early stage, areas and their visual transformation are used in order to explain the "magic" formula $a^2 = b^2 + c^2$. Practical reasons enforce people to objectify an implicit metrical interpretation of the resulting relation. For almost all students, as well as for the majority of educated people today, this interpretation simply interconnects the lengths of the sides of an orthogonal triangle.

This metrical reading of the Pythagorean theorem, which determined and dominated the teaching of geometry through the ages, leads to the computation of distances "absolutely," without any other reference except to positions only. The introduction of coordinates and the use of orthonormal systems of reference were a nontrivial application of the metrical interpretation, as indicated previously. However, we observe that this first use of coordinates does not involve the dimension of time.

1.3 Minkowskian Metric Geometry as Capable to Incorporate Space-Time Concepts

During the nineteenth century, the traditional object of Euclidean geometry was reformed in order to join the newer geometrical conceptions of metrics and invariants under groups of isometric transformations (or rigid motions). A new epistemological view of geometry, as the study of invariants of transformation groups, appeared in F. Klein's *Erlangen Program* (1872). In this embodiment and algebraization, the formulation of Pythagorean theorem was generalized by using several quadratic forms, leading to the concept of Riemannian metric or even to a Minkowskian metric.

In particular, these developments were the basis for the construction of non-Euclidean geometries linking space and time, which had significant applications in physics and could also provide some epistemological implications at the higher levels of education. These mental structures allow new teaching perspectives that could overcome the "timeless" conception of absolute space.

On the following texts, and almost within the mathematical material usually taught in high school, we present in brief a simplified version of Minkowskian geometry as well as its physical interpretation, working in two dimensions (2D); in fact, this is not a limitation from a mathematical point of view (cf. Sartori 1996, chapter 5), although reducing space in one dimension seems to create an additional difficulty for students. Thus, we have the following:

a) The underlying space for both 2D Euclidean and Minkowski geometry is the plane of perception, provided with the Cartesian coordinates (x, y).
b) Two-dimensional Minkowski geometry is based on the (pseudo-Euclidean) quadratic form $H(x, y) = x^2 - y^2$ (while Euclidean geometry is based on the quadratic form $G(x, y) = x^2 + y^2$). The set of points (x, y) with $H(x, y) = 1$ determines a hyperbola which plays the role of the *unitary circle* in this geometry (see Fig. 1).

c) An important example of isometric transformations in 2D Minkowski geometry is the linear transformations $x' = \alpha x + \beta y$ and $y' = \beta x + \alpha y$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 - \beta^2 = 1$, which are called *hyperbolic rotations*. It is notable that the area of any rectangle remains invariant under these transformations. Taking into account these facts, it can be easily obtained (Yaglom 1979, pp. 179–180) that the formula of "distance" between two vectors $\boldsymbol{x} = (x_1, y_1)$ and $\boldsymbol{y} = (x_2, y_2)$ in the Minkowskian plane is:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\left|(x_2 - x_1)^2 - (y_2 - y_1)^2\right|} \tag{1}$$

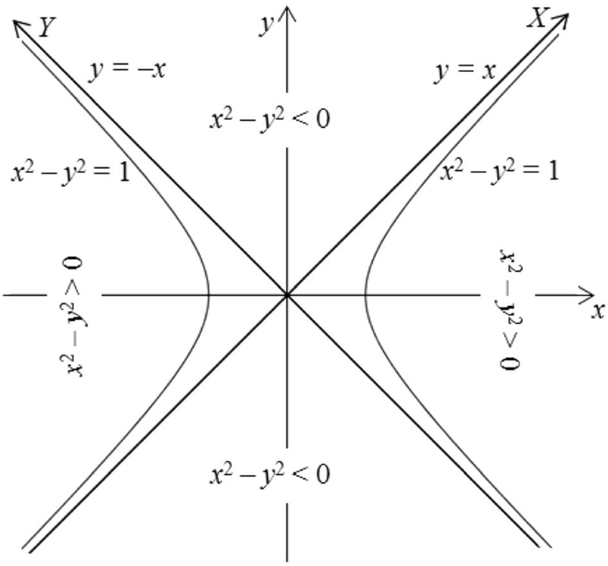


Fig. 1 A physical interpretation of Minkowskian geometry

This metric is equivalent also to:

$$d(\boldsymbol{X}, \boldsymbol{Y}) = \sqrt{2|(X_2 - X_1)\cdot(Y_2 - Y_1)|} \tag{2}$$

where $\boldsymbol{X} = (X_1, Y_1)$ and $\boldsymbol{Y} = (X_2, Y_2)$ are the 45° rotation of $\boldsymbol{x}$ and $\boldsymbol{y}$.

d) Now, we can be led to the concept of (2D) space-time by making use of an informal presentation of Special Relativity by Einstein 1916/1962. Indeed, this interpretation is achieved if we set $y = ct$, where t is the time and c is the speed of light, which we demand to be constant. With an appropriate choice of the constants α, β imposed by the physical constraints and by setting $|u| < c = 1$, we get the following well-known form of Lorentz transformations:

$$x' = \frac{1}{\sqrt{1-u^2}}x - \frac{u}{\sqrt{1-u^2}}y, \quad y' = -\frac{u}{\sqrt{1-u^2}}x + \frac{1}{\sqrt{1-u^2}}y.$$

The physical consequence of these fundamental transformations is *length contraction* and *time dilation* of a body moving at high speed (near the speed of light).

1.4 Reviewing the Literature: Historical Perspectives, Popular Presentations, Epistemological Dialogs, and Teaching Scenarios

It is widely accepted today that in order to understand *modern* mathematical concepts, we have to think of their historical evolution. Barbin (1991, pp. 12–13) has stressed the importance of texts other than Euclid's *Elements* in the teaching and understanding of geometry. She compared Euclid's approach with that of Clairaut, in his *Elements of geometry* of 1765, which did not have a deductive structure and aimed "to give to the reader the mind of invention, the possibility of progressing creatively himself." Moreover, we need to consider Mathematics in interaction with other scientific fields and especially Physics. One of the prominent subjects of presentation, in this direction, is Special Relativity and its mathematical apparatus. This subject gives the opportunity to combine advanced concepts of Mathematics and Physics in an interdisciplinary approach. Therefore, after the wide acceptance of relativistic views, there were many attempts of popularization (cf. Durell 1956) and simplification of the underlying mathematical structure, especially of the interconnection between the Lorentz transformations and the Minkowskian metric. A remarkable example is Einstein's own popularizing attempts (Einstein 1916/1962, pp. 115–122; Einstein and Infeld 1938). On the following texts, we review some distinguished efforts of popularizing relativistic ideas and Minkowskian geometry.

1) Eddington's well-known book (1920) begins with an introductory dialog between three figures, a classical "experimental physicist," a "relativist," and a "mathematician." The character of the "*mathematician*" seems to be rather a marginal one, since the real conflict or interchange of ideas happens between the old-minded (Newtonian) "experimental physicist" and the "relativist." Moreover—and this is most important from our viewpoint—this character seems not to have been thoroughly elaborated. The "mathematician" echoes, more or less, a formalist or conventionalist view of geometry without much care of the physical meaning of geometrical concepts. The role of a creative mathematician in modern times—and correspondingly, that of a teacher of Mathematics today—is inquiry-oriented and far more complex than that of a formalist's caricature, and it is such a role that we wish to portrait to our students by the character of *Space Traveller* (see following texts).
2) It was at the end of the 1930s that physicist G. Gamow's first book was released. This book was the first one of a series in which *Mr. Tompkins* was its main character. In this series, Mr. Tompkins enters "strange" worlds through his dreams, where usual physical laws do not apply. In this way, Gamow aimed at presenting and explaining the modern scientific theories to the general public. In *Mr. Tompkins in Wonderland* (1939), we are acquainted with his "extraordinary voyage," where while he is riding a bike, he crosses the streets of a town at high speed (near the speed of light) and notices that in his direction of motion, all surroundings are contracted. On the other hand, the residents of the town who are at a standstill see the cyclist being contracted. However, today, we are aware of the fact that such naive illustrations do not correspond to scientific thinking and have more or less a metaphorical meaning.
3) During the sixties, an ingenious introduction to space-time physics was written by Taylor and Wheeler (1963). This book is far beyond our didactical intentions and scope, since it is addressed to students of physics with an advanced mathematical background. For example, in answering the question whether there is a "combination of coordinates for two events that will have the same value in the laboratory and rocket frames," it is assumed that the length of intervals is given by the formula:

$$(interval)^2 = (\Delta t)^2 - (\Delta x)^2.$$

After that, the book discusses the invariance of the formula under a suitable set of transformations, a discussion leading to Lorentz transformations by using the ideas of *covariant transformation* and *frame of reference* (Taylor and Wheeler op. cit. pp. 24–58).

4) A newer presentation of Minkowskian geometry was carried out by Yaglom (1979), who mainly attempted "to familiarize future teachers of mathematics … with a geometry different from the Euclidean geometry which they know so well," as it is explicitly announced in the author's preface. This is more or less a clever, algebraically originating presentation realized upon the introduction of specific invariants. It is a systematic study and presentation of geometrical ideas from Galileo to Einstein, clarifying the notion of metric (Euclidean or not), as well as the metric preserving transformations, by using a space of two dimensions only. However, this proposal is not satisfactorily elaborated from a didactical point of view, especially considering high school students.
5) A recent proposal for introducing Minkowskian geometry, which combines the use of dynamic geometry software and popular literature, is that of Felsager's (2004a, b). In particular, his aim was to describe how software "can be used to give a non-axiomatic approach to teaching the non-Euclidean Minkowski Geometry" (Niss 2008, p. 333). In his teaching scenario, Felsager borrowed characters from Lewis Carroll in order to illustrate the similarities and the differences between the Euclidean and the Minkowskian geometry. We observe though that what lies in the kernel of this pedagogical approach is the spirit that "everything is possible in Wonderland." In this context, the Minkowskian metric is introduced in a simplified form due to Yaglom (1979, p. 180). However, this metric is imposed on students from the beginning by using Formula (2) of "Section 1.3", and then, the students are asked to "discover" its properties through representation and transformation of areas in a dynamic geometry environment in order to achieve the quadratic character of the metric.

1.5 Our Didactical and Epistemological Approach

Our didactical approach is interdisciplinary, critical, inquiry-based, and historically inspired. It is *critical and inquiry-based* in the sense that instruction should question the objectification of naively interpreted everyday experience. Inquiry instruction has been examined in recent didactical researches on teachers' practices (for example, cf. Zeichner 1983; Crawford 2000; Chapman and Heater 2010), but this direction is not yet thoroughly explored in geometry lessons (see however van den Brink 1993, 1995; also Kaisari and Patronis 2010). Although our research has an interdisciplinary character, since it combines mathematics and (science) education and borrows elements from literature and history of science, its place is rather within mathematics education. Our main research aim, indeed, is not to teach Special Relativity Theory, neither to introduce students into Minkowski's theory of space-time in a systematic way; *our research aims to explore high school students' conceptions about measurement of length and time, in connection with an inquiry-based reconsideration of geometry.*

More specifically, *our research question* has two interrelated components: (a) whether, and by what strategies, is it possible for students to use their school knowledge to work out a fictional scenario in order to formulate a mathematical model for motions and measurement in space-time; and (b) whether, in this process, the students can reconsider their "timeless" conceptions of motion as reported in "Section 1.2." Our purpose was to lead our students to such reconsideration, because we think that the teaching of geometry and ideas of modern physics are not sufficiently related in high school and that our research could offer such an opportunity.

Historical sources (papers and books addressed to scientists) concerning Minkowskian geometry and relativity theory are usually difficult for high school students to grasp, although, as we have seen in the preceding discussion, interesting popularization attempts have appeared. Instead of using an original historical source as a text for students (see e.g. Jahnke et al. 2002), we presented to them a fictional scenario written by ourselves, which combines characters from J. Vern and H. G. Wells. This scenario was intended to lead the students to a simplified form of the Minkowskian metric, by working out some fictional numerical data. We did not impose a mathematical model to the students to deal with, but we helped them to construct it by themselves. Our didactical choice aimed to question the established epistemological conception of geometry as an a priori system of knowledge, since a *metric* in Riemann's or Minkowski's sense could provide a "physical" meaning to mathematical concepts taught at school. Our empirical findings turned out to justify, at least to some extent, the previously mentioned didactical choices. For example, one of the students claimed that *there is one geometry and in each case there is a different formula* (referring to the metric).

We consider our research as exploring the *boundaries* of what *might be possible* for upper secondary students to achieve in constructing a metric in space-time, after being given a carefully designed scenario. Focusing on students' "possibilities," it seems relevant to perform a strictly qualitative study and analyze some limited episodes with a restricted number of students.

In our approach, geometrical principles are neither limited to physics nor considered as a priori true. Geometry here is viewed as *a context of discussion and inquiry* by questioning the following:

- the role of axioms
- the practice of measurement, and
- the existence and meaning of invariants.

We have already discussed about axioms and measurement. Concerning *invariants*, we observe that aspects of invariance are involved from the very beginning in students' experience, although attention is not usually paid to them in teaching (think for instance the sum of angles in Euclidean triangles). As we pointed out in "Section 1.3," Klein's Erlangen Program was a crucial step towards an entirely new view of geometry, in close relationship with problems of physics. In this setting, invariants are directly related to those transformations which determine the geometry. As an outcome of Klein's viewpoint, the existence becomes possible of more than one geometries on the same underlying region, each geometry possessing its own group of transformations and invariants. In particular, Minkowskian geometry is the study of properties and functions of the coordinates of space-time which remain invariant under the Lorentz transformations. Of course, we admit that high school is not the place of such a complete re-organization of geometrical knowledge. However, we adopt a transitional view, namely that students could possibly be led to mathematize (fictional) experimental data by an inquiry-oriented approach, towards changing the meaning of the coordinates and rediscovering an invariant form.

2 Description of the Project and the Scenario Used

2.1 Elements from the School Curricula

In our project, because of the first component of our research question (see "Section 1.5"), we had to take seriously into account the existing curriculum and knowledge offered at the Greek High School.

According to the Greek educational system, all 10-grade students are taught algebra and Euclidean geometry. The curriculum of algebra aims to teach elementary notions (operations with numbers, equations, inequalities), functions, and graphs. In geometry, the students are taught the basic geometric figures, congruence of triangles, symmetry, and parallelograms. Euclidean geometry is presented not as a model of physical space but as a closed deductive theory, and the discussion of Euclid's 5th postulate (including the possibility of other geometries) is nearly neglected. On the other hand, the physics curriculum refers mainly to rectilinear motion (uniform and uniformly accelerated), as well as Newton's laws and their implications at an elementary level. Experiments and diagrammatic representations are nearly absent, except for some expositional activities.

Algebra for 11th-grade students comprises systems of equations, monotonicity of functions, and some elements from trigonometry. Inscribed polygons, similarity of figures, Pythagoras' theorem, metric relations, and areas are taught in geometry. At this stage, students may choose either the natural sciences, or technology, or the social studies direction. Students who will attend natural sciences and technology have some extra hours in analytic geometry (vectors, lines, conic sections), while in physics, they are taught curvilinear motions and horizontal projectile motions without any use of differential calculus.

Twelfth-grade students of natural science and technology direction are furthermore taught the elements of differential and integral calculus, while in physics, they are taught oscillations, waves, and rigid body mechanics. It is remarkable that there is no interconnection between mathematics and physics curricula, a fact which does not help teaching and students' understanding of the interdisciplinary character of the subjects taught.

2.2 Preliminary Phase and the Participating Students

During the school year 2013–2014, students in a town near Athens attending the 11th and 12th grades were invited to participate voluntarily in a project about mathematics and science. Sixteen out of about a hundred students of this town accepted the invitation to participate in our research. They were given a text that was a part of our *scenario* which is presented in "Section 2.3." The text and the questions following it aimed to guide students to think about *motion* and the need for a new *metric* in a space-time of two dimensions (x, t). The students were asked to study the text and write down their thoughts about it. Then, an individual interview took place for about one hour, in which each student was asked some general questions regarding the following: (a) science fiction in popular literature; (b) history of science; and (c) the role of axioms in geometry. The following questions are examples of each category:

(a) Have you read any novels of Jules Verne and Herbert George Wells, e.g. *The Time Machine*?
(b) What do you know about Albert Einstein's work and what is the source of your information (school, friends, popular books, media)? Have you heard of the Special Relativity Theory?
(c) What do you think is the role of axioms in geometry? Can you mention any axiom of Euclidean geometry?

This preliminary phase was necessary for us, in order to ascertain the extent of engagement, the degree of knowledge, and the interests of each student, so that we could form a final group for the project. Four of the students, two boys (let us call them *Alexander* and *Constantinos*) and two girls (let us call them *Helen* and *Maria*) showed a distinctive interest and engagement and formed *the main group of participants*. We needed such an interest and engagement on the participant students' part, because our intention was to explore the possibilities of students at certain depth. Our preliminary phase and formation of the final group of participant students are similar to those of Kaisari and Patronis (2010), in which, however, the final group of students was selected according to their particular engagement in responding to a narrative text.

We proceeded to the *main phase* of our research (to be described in "Section 3") with the four students, who were of different educational/cultural levels, with various extra-curricular activities, and different family statuses. In this sense, this group of students cannot be considered as "exceptional." More specifically,

- *Alexander* was a 12-grade student who would enroll in technological studies after school, came from an average educated background, and had no particular interests.
- *Constantinos* was also a 12-grade student who would enroll in technological studies after school, and whose parents were well-educated. He was spending long hours doing sports, while his only extra-curricular reading had to do with health and nutrition magazines.
- *Helen* was an 11-grade student who would study social studies and raised by well-educated and prosperous parents. She was cultivated, sensitive, imaginative, and interested in dance and theater.
- *Maria* was an 11-grade student who would study natural sciences. She was a good and imaginative thinker although there were no educational stimuli from her family environment. She was interested in painting, and she read only her school books.

All four students had meager knowledge of literature related to science or science fiction. In addition, what they knew about the life and work of great scientists like Galileo, Newton, and Einstein was rather popular myths like "Galileo's experiments made from the campanile of Pisa" or "Newton's apple" (cf. Numbers and Kampourakis 2015).

Helen and Maria knew about inversely proportional quantities, while Alexander and Constantinos were, moreover, in acquaintance with both forms of the equation of hyperbola taught at school i.e. $y = \frac{c}{x}$ and $\frac{y^2}{a^2} - \frac{y^2}{a^2} = 1$. However, they did not know how these forms could be related to each other.

As far as geometric axioms are concerned, the students had heard of them at school, but they were completely unaware of their role in the foundation of geometry. During the preliminary phase, the researcher gave them two examples of axioms in Euclidean geometry: *It is possible to draw a unique straight line from any point to any other point*, and *it is possible to extend a finite straight line continuously*. Even after these examples, the students continued to confuse axioms with unquestionable knowledge or familiar theorems.

2.3 The Fictional Scenario and Its Rationale

In our research, we based on a fictional scenario written by ourselves, which combines characters from J. Verne's *De la Terre à la Lune* (1865) (President Barbicane, J. T. Maston, Captain Nicholl, Michel Ardan) together with the character of "Time Traveller" in H. G. Wells' *Time Machine* (1895), who appears as "*Space Traveller*" in our text, adjusted to a new situation. We needed this name in order to put emphasis mostly on the notion of *space* rather than that of *time*, since we did not wish to enter into questions about the nature of time and its physical or philosophical significance. Our choice takes into account the fact that, as J. Verne himself used to say, his own stories were based on the actual scientific progress of his time, while the younger and talented H. G. Wells was more concerned about society in the distant future. Also, all characters of our scenario pose reasonable questions on mathematics and natural sciences and arguments to support their views.

Our text has been created with an effort to produce lively dialogs on scientific questions and an atmosphere of inquiry, contemplation, and dispute. In this way, we avoided presenting scientific ideas to the readers in a "magical" way, as e.g. in the context of *Alice's adventures in Wonderland*.

Our scenario consisted of two parts: In part I, the *Space Traveller* appeared as a young scientist who tried to verify his suspicion that Euclidean geometry cannot be the only possible geometry in physical space. The Space Traveller has access to data gathered during the travel of *Columbiad* spaceship, which concerns the following:

- the measurement of the *length* (l) of the spaceship, and
- the *time* (t) between two instantaneous gleams of a light source placed on the spaceship.

These data (see Table 1) have been collected separately by two groups of observers: those being in the spaceship and traveling at a speed comparable to the speed of light ("moving" observers), and those staying on the Earth ("fixed" observers). What takes place in part II is the mathematical treatment of the facts in part I. Furthermore, explanations about the underlying Mathematics are given to the students.

Part II is the sequel of part I, which was written by taking into account the students' responses. This part leads a mathematical elaboration and in particular the transformation of the Minkowskian metric to its quadratic form (see "Section 1.3").

On the following discussion, we refer only to part I of our scenario, which is relevant to the episodes that we wish to analyze. The text of this part of the scenario is followed by a number of questions to be answered by students. Some of these questions are as follows:

Question 1. Notice the data in Table 1 about the "fixed" observers[1] and try to think of a relation connecting the spaceship length l and the time t between two gleams of light, as the velocity v of the spaceship changes:

Question 2. In an orthogonal system of coordinates place the points with coordinates (l, t) of the previous table. What can you notice? What kind of relation exists between the values of l and those of t? Is your first estimation, which you formed out of Question 1, confirmed or not?

Question 3. Well, the astronauts ("moving" observers) find out that the length as well as the time span remain unchangeable, but in contrast for the "fixed" observers these magnitudes bear a change. Finally, who's right? What is valid in reality? In your opinion, where is this differentiation attributed to?

On the following, the length l of the spaceship was symbolized with x and the length of the time span t which takes place between the two gleams, with y. So, the use of kinematics' terms is avoided, while at the same time, the same unit of measurement is attributed to these

[1] The students were given a table with 24 lines. Here, we present an indicative part of this table.

Table 1 Length, time, and velocity values

v	l	t
0.2	35.26	1021
0.33	33,994	1059
0.4	33	1091
0.553	30	1.2
0.6	28.8	1.25
0.745	24	1.5
0.832	20	1.8
0.866	18	2
0.89	16,415	2193
0.943	12	3
0.96	10	3.6
0.968	9	4

magnitudes (x and y). This unit becomes a unit of length if we place t with $y = c{\cdot}t$, so y will be given in $\frac{\mathrm{m}}{\mathrm{s}}{\cdot}\mathrm{s} = \mathrm{m}$. As a result, the graph of Question 2 remains the same with the only difference the renaming of the axes from (l, t) to (x, y), as follows (Fig. 2):

Parallel line segments were drawn from each point of the graph towards the semi-axes Ox and Oy which form along with the semi-axes rectangular parallelograms.

Question 4. What do all these rectangles have in common?

A final question was added about a controversy which was supposed to occur involving the characters of the scenario, on the rectangles formed in the fourth question: *Can these rectangles be considered as actually one and the same rectangle, in the sense that the various shapes appearing may be due to a kind of* "*motion*" *of the rectangle in the given coordinate system*? The Space Traveller made here a conjecture, advocating a view of geometry as the study of properties of figures remaining invariant under certain transformations. Here, an

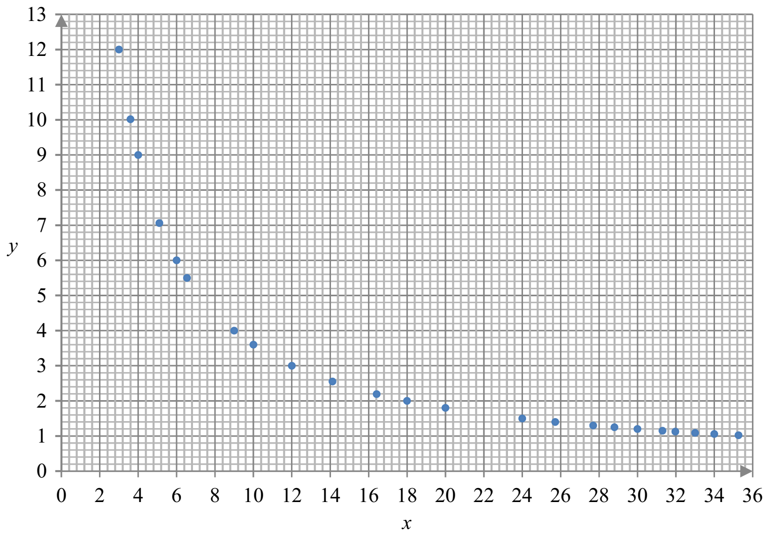


Fig. 2 The obtained result in plotting a graph based from Question 2 with the axes (l, t) renamed to (x, y)

important such property is the *area* of rectangles. Thus, the characters of the scenario are on the threshold of a major breakthrough of a new geometry on the plane.

Question 5. Can you find a "tape measure" according to which the length of the diagonals of the rectangles can always remain the same?

We need here to justify our choice of providing the students with a *fictional* table of data, instead of real ones. Because of their large-scale nature, such data are not possible to be achieved in a laboratory. Moreover, with our fictional data, we did not attempt to lead the students to discover a *physical law* (such as length contraction and time dilation). Our point of view is that of a *geometrical* rather than *physical* inquiry; we do not claim that our scenario, as well as our fictional data, can serve as an introduction of young students to Special Relativity Theory. Nevertheless, our fictional data are not completely arbitrary, but they are in accordance with certain predictions and thought experiments in the early development of Special Relativity (see e.g. Einstein op. cit.).

2.4 Planning the Research and Style of Interaction of the Researcher with the Students

Concerning the role of the researcher, there was a clear-cut decision that he would not guide the students but he would do the following:

- ask clear questions in order to understand students' thinking (see e.g. line 24 of Episode I);
- direct students' attention to the questions of the text (see e.g. line 4 of Episode I);
- encourage students to clarify their own reasoning (see e.g. line 28 of Episode I or line 36 of Episode II); and
- confirm some answers with significant meaning for the progress of the conversation (see e.g. line 18 of Episode I) and not reject any irrelevant answers, but discuss them properly (see e.g. line 9 of Episode I in "Section 4").

From this specific choice of the style of interaction between the researcher and the students, the overall planning of the process could not have been entirely predicted in advance. Besides, a scenario of this type is planned to facilitate some possible learning trajectory, without this path being unique.

3 Overview of the Process and Collection of Empirical Data

During the main phase of our research, Alexander and Constantinos worked together, as well as Helen and Maria. Thus, two constant couples of participants entered into corresponding *focused interviews* (Stewart and Shamdasani 1990). Each couple was given the text of the scenario, pencils, graph papers, calculators, rulers, compasses, and a PC in which *MS Office* and *GeoGebra* had been installed. There was also a whiteboard, on which the researcher would quite often write something. All dialogs between students and the researcher or between students themselves were audio-recorded and video-taped. The main phase of the project was divided into two meetings, at a place separate from school, with about 15 days difference between the meetings.

On the first meeting, the students were given part I of our scenario (as in "Section 2.3"). Firstly, we gave them the text and then the questions one after the other. It took them about one hour and a half to answer the questions. What puzzled all students most was Question 3, actually which of the two groups of observers was right and what is really valid. At the same time though, this puzzling question revealed a wealth of ideas regarding the natural world. Another difficulty, common among the students, was the employment of a non-orthonormal system of axes in Question 2. Contrary to the boys, Helen and Maria did not manage to answer Question 5 until the researcher explained it to them over their second meeting, thus creating a "link" between the two meetings.

The findings of the first meeting as analyzed and discussed in "Section 4" and "Section 5" together with the participants' questions led us to the second meeting. We came up with a sequel of part I and some new characters entering the scenario. Computer-aided activities were also used in this part, aiming at the students' own *experimentation* and the expression of their *conjectures*. The students were informally introduced to the notions of "event" and of "space-time." The discussion of Minkowskian metric, in this second meeting, provided also the opportunity to encourage the students to respond to a question of their own and discover how the two familiar forms of the equation of a hyperbola, namely $y = \frac{1}{x}$ and $y^2 - x^2 = \left(\sqrt{2}\right)^2$, are related to each other.

4 Analysis of the Two Episodes

On the following texts, we analyze two episodes of the first part of our research. These episodes are exemplary, because their nature is neither deductive, nor purely inductive. Indeed, our scenario, presenting fictional data, is rather closer to Einstein's own *thought experiments* (cf. Einstein 1916/1962 e.g. chapter 9).

4.1 "Determining Moments": An Interpretive Framework of Analysis

Each "episode" analyzed subsequently is a relatively self-contained part of the dialog, characterized by a particular theme. We also divided each episode into several "scenes," which exhibit a unity in students' responses and explanations. In each episode, we had identified certain "*determining moments*." By this term, we mean those short parts of the dialog, from which a change of students' perception of the subject, or at least a questioning of preexisting beliefs, can be deduced from their reactions or verbal responses. More exactly, a determining moment is a short period with an external appearance of questioning and re-arrangement of former knowledge; this period may, or may not, be preceded by internal cognitive conflict in the thinking person(s). Thus, every determining moment gets its meaning from the episode to which it belongs or from previous episodes, and in turn, it affects the next steps of the interview.

As a parallel setting in the history of mathematical ideas, we can say that a "determining moment" is a historical phase in which people reflect on the epistemic function of their activity and try to mathematize it. This reflection and mathematization usually takes place in a period of ambiguity and indecision, or in a period of reorganization. As a typical example of a determining moment at a high level of abstraction, we consider the *Erlangen Program*, which has been already discussed in our introduction (cf. "Section 1.4"). However, this was an exceptionally fortunate case in which a determining moment in mathematics was fully realized and explicitly expressed by the mathematicians experiencing it. As a contrast to this happy case, for example, Descartes and Fermat did not realize that, by changing the meaning of “coordinates” in Apollonius’ works, they had invented modern analytical geometry (cf. J. Klein 1985). Thus, it becomes clear that determining moments in history of mathematics often brings about a silent change, which is generally different from what is described by Kuhn (1962) as *paradigm shift*.

Moreover, other examples of determining moments from history of mathematics, such as the introduction of negative or complex numbers in solving polynomial equations, as well as the use of infinitesimals, show some uneasiness and lack of certainty (Patronis and Spanos 2013). Similarly, in our study, there are determining moments (not always recognizable by students) in which students realize that “something has to change” in their strategy, and they may think of a first rough plan, but they do not always come to an effective strategy in order to perform this plan.

4.2 Episode I: Searching for the Relation Between Length (*l*) and Time (*t*)

In this episode, Maria and Helen are trying to work out Question 1 (see “Section 2.3”). The reason why we chose to present this specific episode is that the students here went through five different stages until they get some definitive answers, proceeding to a quantitative interpretation of the problem.

Scene 1: Numerical operations

1. Helen: Well, what I can see is that the more l decreases, the more t increases. I have no idea if there is a pattern for that … Should I have been aware of that?
2. Researcher: Hmm …
3. Helen: Are there other things we should notice?
4. Researcher: Are you fully aware of what the question asks you to do?
5. Maria: Yes, to find out what is happening with l and t. There is a relation between them … I think when this is 18, that is 2 (these numbers are shown in Table 1). Then, the half of 18 is equal to the square of the next number (that is two squared). Just as it happens with the half of 9 …
6. Helen: Four squared is not equal to …
7. Maria: Ah, yes that’s right.

Helen could definitely conceive a relation between the variance of length and that of time, but in naive “qualitative” terms. Maria, who could similarly conceive the relation of these magnitudes, tried to reach an imaginative, complicated relation instead of noticing the obvious (that is $l{\cdot}t$ is constant).

8. Helen: The fact that l decreases and t increases is not a sufficient explanation? Is it far too general?
9. Researcher: I would say that it is “qualitative.” What we are looking for is something “quantitative.”
10. Helen: Look, there are no negative numbers! (She refers to the scenario given.)

Scene 2: Change of context

After a few moments of silence, Helen makes a suggestion:

11. Helen: Ok, what I do remember from school about Physics is that velocity is equal to the distance a body travels over the time it takes.

12. Researcher: Yes.
13. Helen: If we use this formula $v = \frac{l}{t}$ what will the outcome be? Something like a common result? What if, when we give different numerical values …

After a few fruitless attempts, the researcher explains that this formula expresses the average velocity of a body. The data of Table 1, though, do not refer to the distance a body travels but to its length.

Scene 3: Focusing on the problem
14. Researcher: … What we have, is an object travelling.
15. Helen: Yes.
16. Researcher: And while it's travelling, its length…
17. Helen: Gets smaller.
18. Researcher: That's right.
19. Helen: At least this is what we can read on the screen of the meter.
20. Maria: Yes.
21. Helen: Ok, every time I see a fast car, for example a Ferrari, running at 280 km/h, I see its figure being distorted; I never manage to see the whole car.

    Helen's interpretation (line 21) was based on common sense and has to do with the limits of human optical perception.

Scene 4: A "black hole"
22. Helen: I don't know … Is it possible that this spaceship enters 'somewhere' and disappears, and the observers can see only the part which hasn't been inserted in this "somewhere"?
23. Maria: Where can it enter?
24. Researcher: Where exactly?
25. Helen: In a black hole... I don't know. There are so many strange things in the space!

The use of terms from everyday language in order to describe objects and phenomena of the macrocosm (i.e. *white dwarf*, *red giant*, *black hole*) has been connected to *anthropomorphic mental images* (Al-Balushi 2013). Students often believe that the heavenly bodies have unexpected properties, which cannot be predicted or described according to physical laws. Within this context, Helen (line 25) imagined a *black hole* as a cave hidden among other strange celestial bodies, which "swallows slowly" the spacecraft.

Scene 5: A determining moment
26. Maria: It [the spaceship] runs *faster*.
27. Helen: Because of the *speed* ...
28. Researcher: Go on ...
29. Helen: Is it because the time and the length are things that *we* have defined *according to our experience*, let's say, *by not taking into account what really happens outside the Earth's environment*. Perhaps in universe the same things don't apply, and there are different laws out there ...
30. Researcher: What were you going to say about the speed?
31. Helen: *Is it possible that the spaceship is compressed by the high speed*? But *how is it possible* that some people measure the length as equal to 36 m and others as equal to a smaller one?

The situation here has exceeded the obstacle of visual perception that occupied the girls before. There is a strong analogy with the scientific notion of *length contraction*, although the girls referred only to the length of the spaceship (line 31). The girls inferred intuitively that the spaceship is possibly compressed because of its "high speed." Helen's expression refers indeed to *speed* and not e.g. to *acceleration* because the students followed the lines of our "story" in which measurements are not produced at the beginning of the travel of the spaceship, but later, when its speed is constant. Therefore, what we have here is a *determining moment* for both students, Maria (line 26) and mainly Helen (lines 29 and 31). Newtonian beliefs were questioned, together with the whole mental model of a homogeneous and isotropic physical space with universal laws. Indeed, the girls asked themselves (line 29) whether the natural laws governing the Earth and its environment apply also to the remotest parts of the universe. However, Helen's words in line 31 are expressing a conflict that takes place inside her.

4.3 Episode II: Searching for a Suitable Metric

Scene 1: Applying the Pythagorean theorem

In this episode, we can see Constantinos' and Alexander's attempt to answer Question 5. A crucial step towards the answer to this question is to find a new suitable *metric*, such as that the diagonals of all the rectangles of Question 5 are found to have the same length.

Initially, the students tried to apply the Pythagorean theorem to the triangles formed by the sides and the diagonals of the rectangles. However, they soon realized that it cannot be applied in this situation, as it gives incorrect results (actually the metric is Minkowskian, see "Section 1.3"). At the same time, the researcher explained to them that the Pythagorean theorem is equivalent to the Euclidean form of the metric (as in "Section 1.4").

Constantinos, trying to formulate the new metric, asked a question which is related to the number of dimensions of space:

32. Constantinos: Does this apply, I mean Euclidean Geometry, in one plane? And this new tape measure applies to a three-dimensional or to one …
33. Researcher: With more dimensions?
34. Constantinos: Yes.

Scene 2: Invariance of area as an axiom—a determining moment?

It was necessary for the researcher to intervene at this point, by recommending that the students should take Question 4 into consideration. In this way, the students were led to the question if that was the real "tape measure" they were looking for, since there was not any kind of verification.

35. Constantinos: How about the formula we have already found? Isn't it enough?
36. Researcher: Go ahead.
37. Alexander: The one in which the area is equal to 36.
38. Constantinos: $l{\cdot}t$.
39. Alexander: That's the only way. This multiplied by this is equal to that multiplied by that and that multiplied by that (referring to the sides of the rectangles under question).
40. Researcher: You have a point here. But here we've got x and y.
41. Alexander: Ok. The formula will be $x{\cdot}y = OM$. But we don't know if that is so. How do we know that it's actually OM and it's not something else?
42. Constantinos: So here we set an "axiom" or not?

The students discovered a pattern using the formula for the area of a rectangle as a mathematical tool potentially leading to an invariant magnitude of the measurements. Alexander announced this verbally in a clear, although unusual way (line 39). Of course, the boys did not seem, at this moment, to perceive the interconnection between the invariant quantity $x{\cdot}y$ and the desired metric.

Evidently Constantinos' question (line 42) has to do with the specific fact that students were led to the formula $OM = x{\cdot}y$ without a justification. This same response shows how students are puzzled in general about the role of axioms in geometry.

Scene 3: A different type of metric in each case

Gradually, Alexander and Constantinos found out the metric in form $OM = \sqrt{x{\cdot}y}$ in question and actually the first part of the main phase of our research is completed.

43. Researcher: So, was the Space Traveller right? It was his belief, his suspicion, you can say it any way you like, that Euclidean Geometry couldn't be the only existing geometry in our surroundings. Was he finally right?
44. Alexander: Probably yes … Therefore there is not only one. There are a lot of Geometries.
45. Constantinos: What I think is that there is *one* geometry and we use it in different cases. In each case there is a different formula [metric].

Constantinos made here a significant remark, rethinking the conjecture of Space Traveller (cf. “Section 2.3”, the ending phrases before Question 5). Although this issue is informally described, Constantinos suggested a kind of *unity* among the geometries, which is implicitly related to the modern notion of *metric* as an additional structure independent of incidence axioms. Thus, we regard this conversation as a *determining moment* for the two boys, which is beginning from line 32 and ending with line 45.

5 Further Discussion and Interpretation—Concluding Remarks

This final section contains a further discussion of dialogs with the two couples of participating students, interpreting their determining moments and more generally their cognitive behavior and strategies.

Daring here an epistemic parallelism, we could say that, at first, Helen's and Maria's strategy is similar to Timaeus' urge in Plato's *Timaeus* 29d as interpreted by Cornford (1937, p. 23): since nature is governed by infinite complexity, our physical explanations cannot easily reach the accuracy of proof. Therefore, at the beginning, we should “accept the *likely story* (εἰκώς μῦθος) and look for nothing further,” until we find a *likely account* (εἰκώς λόγος) i.e. an “account which combines likelihood with necessity” (cf. *Timaeus*, 53d, again interpreted by Cornford 1937, p. 212). In a similar manner, feeling that their potentials were limited in mathematics and willing to give convincing answers to the questions, the girls resorted initially to a naive interpretation and to the popular mythical conception of “black holes.”

The girls' cognitive obstacles may be divided in two categories: those related to the persistence of Euclidean geometry conceptions and those due to their difficulties in the use of algebraic tools. Indeed, in Scene 1, they dealt with the problem in a naive “qualitative” way, while working with numerical operations. But in Scene 2, they changed context, trying to discover an existing “physical law” behind the given data. In Scene 3, they attempted to focus on the question, but they found the new context difficult and they resorted to the popular myth of "black hole" (Scene 4). Then (Scene 5), they intuitively suggested a physical interpretation of the problem. The two girls experienced a determining moment by coming to the intuition that the spaceship's length is possibly "compressed" because of its "high speed." The girls arrived at this explanation without any aid other than intuition and imagination. Their *likely account* consists in referring to the spaceship's high speed in order to provide a possible, non-mythical, explanation of its compression.

The two boys who participated in our research tended to use the mathematics taught at school in a more "systematic" way, at least compared to the girls. This can be explained by their being at an upper grade and by their interest in technology-oriented studies. Alexander and perhaps more neatly Constantinos were seeking mathematical formulas in order to fit the fictional empirical data presented to them, in a way compatible with the school practice, which promotes algebraic operations. Nevertheless, this attitude was perhaps helpful in preparing the boys' determining moments, as analyzed in the preceding texts, because it helped them to look at the situation algebraically. Thus e.g. the constant area of rectangles (in Scene 2 of Episode II) led them to a correct formula nearly defining a new metric in two-dimensional space-time.

The most important development in our research experience was perhaps the determining moment in Scene 3 in which the two boys conceived the notion of metric as an additional structure on the underlying two-dimensional (2D) space. The underlying space is unique in its incidence relations, but it can be equipped with different metrics corresponding to different geometries. In Constantinos' terms, *there is one geometry with a different formula in each case.*

It was the view of the quantity $\sqrt{x{\cdot}y}$ as immediately related to a new metric, that enabled the two boys later (in the second meeting), to investigate the concept of "circle" in the new Minkowski Geometry on a 2D plane. Also, these boys were able, in the second meeting, to discover the equivalence of the two familiar forms of the equation of a hyperbola, namely $y = \frac{1}{x}$ and $y^2 - x^2 = \left(\sqrt{2}\right)^2$. In fact, these formulas express the notion of the "unit circle" in 2D Minkowski Geometry (see "Section 1.3"). However, when we passed, in the second meeting, from the merely geometrical form $OM = \sqrt{x{\cdot}y}$ to its physical interpretation in 2D space-time (by replacing the abstract points (x, y) by events (x, t)), the boys had a difficulty to conceive the hyperbola $x \cdot t = r^2$ as a "circle" in 2D space-time. We did not insist at this point, because, as we have emphasized in "Section 1.5," our research aim was not to teach Special Relativity Theory, but to encourage an inquiry-based reconsideration of Geometry.

Also, the dilation of time was not at all mentioned by the participant students. However, the "compression" of the spaceship's length was related to its high speed by the girls. One could draw a distant parallel between this situation and a respective conception of H. A. Lorentz (1904) who, after the formulation of the transformations leaving Maxwell's equations invariant, regarded time dilation as having no physical meaning: "Lorentz never interpreted the primed variables as anything more than a mathematical construct, an "imaginary system" S' in which the body is formally at rest. The real rest system is S_r and the real time is t in all frames. The parameter t', which Lorentz called "local time" (*Ortszeit*), is *not* the time recorded by a clock in the rest frame of the moving body" (Sartori 1996, p. 124).

In concluding, in our research two main students' strategies were identified. One of them (belonging to the two boys) makes a creative use of preexisting school knowledge in algebra and geometry, in order to fit the given data and provide the required formula; this strategy may stem from common school practice, but fortunately it does not miss completely the physical meaning, as the boys' determining moments show. The other strategy (belonging to the two girls) at first resorts to popular myths and then passes to an explanation relying on imagination and intuition. From the perspective of inquiry-based instruction, if the previously discussed strategies are considered in combination, it can be said that they approached the core of the subject. The significance of these findings appears within an interdisciplinary and historical educational perspective.

Compliance with Ethical Standards

Conflict of Interest The authors declare no conflict of interest.

References

Al-Balushi, S. (2013). The nature of anthropomorphic mental images created by low and high spatial ability students for different astronomical and microscopic scientific topics. *International Journal of Science in Society, 4*(4), 51–63.

Barbin, E. (1991). The reading of original texts: how and why to introduce a historical perspective. *For the Learning of Mathematics, 11*(2), 12–13.

van den Brink, F. J. (1993). Different aspects in designing mathematics education: three examples from the Freudenthal Institute. *Educational Studies in Mathematics, 24*, 35–64.

van den Brink, F. J. (1995). Geometry education in the midst of theories. *For the Learning of Mathematics, 15*(1), 21–28.

Chapman, O., & Heater, B. (2010). Understanding change through a high school mathematics teacher's journey to inquiry-based teaching. *Journal of Mathematics Teacher Education, 13*(6), 445–458.

Cornford, F. M. (1937). *Plato's cosmology: the Timaeus of Plato with a running commentary*. London: Routledge and Kegan Paul.

Crawford, B. A. (2000). Embracing the essence of inquiry: new roles for science teachers. *Journal of Research in Science Teaching, 37*, 916–937.

Driver, R., Guesne, E., & Tiberghien, A. (Eds.). (1985). *Children's ideas in science*. London: Open University Press.

Durell, C. V. (1956). The theory of relativity. In J. Newman (Ed.), *The world of mathematics* (pp. 1107–1143). New York: Dover.

Eckstein, S. G., & Kozhevnikov, M. (1997). Parallelism in the development of children's ideas and the historical development of projectile motion theories. *International Journal of Science Education, 19*(9), 1057–1073.

Eckstein, S. G., & Shemesh, M. (1993). Stage theory of the development of alternative conceptions. *Journal of Research in Science Teaching, 30*, 45–64.

Eddington, A. S. (1920). *Space, time and gravitation*. London: Cambridge University Press.

Einstein, A. (1962). *Relativity: the special and general theory*. London: Methuen & Co Ltd..

Einstein, A., & Infeld, L. (1938). *The evolution of physics*. Cambridge: Cambridge University Press.

Felsager, B. (2004a). Introducing Minkowski geometry using dynamic geometry programs, a short paper for the Topic Study Group 10 at ICME-10, Copenhagen, July 2004.

Felsager, B. (2004b). Through the looking glass: a glimpse of Euclid's twin geometry, Supplementary Notes, ICME-10, Copenhagen, July 2004.

Gamow, G. (1939). *Mr. Tompkins in wonderland*. Cambridge: Cambridge University Press.

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., da Silva, C. M. S., & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education* (Vol. 6, pp. 291–328). New York: Springer.

Kaisari, M., & Patronis, T. (2010). So we decided to call 'straight line' (…): mathematics students' interaction and negotiation of meaning in constructing a model of elliptic geometry. *Education Studies in Mathematics, 75*, 253–269.

Klein, F. (1872). *Vergleichende Betrachtungen über neuere geometrische Forschungen*. Erlangen: Verlag von Andreas Deichert **(in German)**.

Klein, J. (1985). The world of physics and the "natural" world. In R. B. Williamson & E. Zuckerman (Eds.), *Jacob Klein, lectures and essays* (pp. 1–34). Annapolis: St John's College Press.

Klein, F. (2004). *Elementary mathematics from an advanced standpoint: geometry*. New York: Dover.

Kuhn, T. (1962). *The structure of scientific revolutions*. Chicago: The University of Chicago Press.
Lappas, D., & Spyrou, P. (2006). A reading of Euclid's elements as embodied mathematics and its educational implications. *Mediterranean Journal for Research in Mathematics Education, 5*(1), 1–16.
Lorentz, H. A. (1904). Electromagnetic phenomena in a system moving with any velocity smaller than that of light. *Koninklijke Akademie van Wetenschappen te Amsterdam. Section of Sciences. Proceedings, 6*(1903–1904): 809–831.
Matthews, M. R. (1994). *Science teaching: the role of history and philosophy of science*. New York: Routledge.
Niss, M. (Ed.). (2008). Proceedings of the 10th International Congress on Mathematical Education, 4–11 July, 2004. IMFUFA, Department of Science, Systems and Models, Roskilde University, Denmark.
Numbers, R. L., & Kampourakis, K. (Eds.). (2015). *Newton's apple and other myths about science*. Cambridge: Harvard University Press.
Patronis, T. (1994). On students' conception of axioms in school geometry. Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 9–16). Lisbon, Portugal.
Patronis, T., & Spanos, D. (2013). Exemplarity in mathematics education: from a romanticist viewpoint to a modern hermeneutical one. *Science & Education, 22*, 1993–2005.
Poincaré, H. (1952). *Science and hypothesis*. New York: Dover.
Sartori, L. (1996). *Understanding relativity*. California: University of California Press.
Stewart, D. W., & Shamdasani, P. N. (1990). *Focus groups: theory and practice. Applied social research method series (volume 20)*. Newbury Park: Sage Publications.
Taylor, E. F., & Wheeler, J. A. (1963). *Spacetime physics*. San Francisco: Freeman & Co..
Verne, J. (1865). *De la Terre à la Lune*. Paris: P. J. Hetzel et Co (in French).
Wells, H. G. (1895). *The time machine*. London: William Heinemann.
Whitaker, R. J. (1983). Aristotle is not dead: student understanding of trajectory motion. *American Journal of Physics, 51*, 352–357.
Yaglom, I. M. (1979). *A simple non-Euclidean geometry and its physical basis*. New York: Springer-Verlag.
Zeichner, K. M. (1983). Alternative paradigms of teacher education. *Journal of Teacher Education, 34*, 3–9.